Problem Set 6 : Due Thursday, February 23

See the course website for homework policy.

1. Show that there is no simple group of order 84.

2. Let G be $C_{15} \rtimes C_2^2$, where the generators of C_2^2 act by 4 and 11. Let $P \subset G$ be the image of a splitting $G \leftarrow C_2^2$.

   (a) Recall that the normalizer of P is defined to be $N_G(P) := \{g \in G : gPg^{-1} = P\}$. Compute the order of $N_G(P)$.

   (b) How many 2-Sylows does G have?

3. Show that there are no simple group of order 300.

4. Let G be a simple group of order 360. How many 5-Sylows does G have?

5. In this problem, we will check that A_n is simple for $n \geq 5$. Our proof is by induction on n, and you may assume the base case $n = 5$ without proof. (Alternatively, here is a slick proof: The conjugacy classes in A_5 have sizes 1, 15, 20, 12 and 12. No subset of these adds up to a divisor of 60, so no union of conjugacy classes can be a subgroup.)

   (a) Let $n \geq 5$ and let σ be any nontrivial element of A_n. Show that there is some σ', conjugate to σ, such that $\sigma \neq \sigma'$ and $\sigma(\sigma')^{-1}$ has a fixed point.

   Let $n \geq 5$ and let N be a nontrivial normal subgroup of A_n. For $1 \leq i \leq n$, let H_i be the subgroup of $\sigma \in A_{n-1}$ with $\sigma(i) = i$.

   (b) Show that $N \cap H_i$ is nontrivial for some i.

   (c) Show that $N \supseteq H_i$ for some i. Then show $N \supseteq \bigcup_i H_i$.

   (d) Show that $H = A_n$.