Problem Set 8 : Due Thursday, March 23

See the course website for homework policy.

On this problem set, you may assume the truth of the main Theorem of Galois theory: Let L/K be a finite Galois extension, with Galois group G. The fields F with $K \subseteq F \subseteq L$ are in bijection with the subgroups H of G. The bijection takes $F \mapsto \{g \in G : g\theta = \theta \ \forall \theta \in F\}$ and $H \mapsto \{\theta \in L : g\theta = \theta \ \forall g \in G\}$. We have $[L : L^H] = \#(H)$.

1. Suppose that $K/\mathbb{Q}$ is a Galois extension. Prove that, for any subfield F of K, there exist subfields $F_1, \ldots, F_r$ of K such that $F = \bigcap F_i$ and all the degrees $[K : F_i]$ are prime powers.

2. Let K be the field $\mathbb{Q}(\omega)$, where ω is a primitive 3rd root of unity. Write $\alpha \mapsto \bar{\alpha}$ for the nontrivial automorphism of K. Let $L = K(\sqrt[3]{\alpha}, \sqrt[3]{\bar{\alpha}})$ for some $\alpha \in K$.
   (a) Show that $L/\mathbb{Q}$ is Galois. Show that $\#\mathrm{Gal}(L/\mathbb{Q})$ divides 18 and that $\#\mathrm{Gal}(L/\mathbb{Q})$ is even.
   (b) Show that $\mathrm{Gal}(L/\mathbb{Q})$ is a subgroup of S_3 if and only if $\alpha/\bar{\alpha}$ is a cube in K.
   (c) Show that $\mathrm{Gal}(L/\mathbb{Q})$ is a subgroup of $\mathbb{Z}/6$ if and only if $\alpha\bar{\alpha}$ is a cube in K.
   (d) Let $\alpha = 7(2 - \omega)$. Note that $\alpha\bar{\alpha} = 7^3$. Explicitly write out how an order two element of $\mathrm{Gal}(L/\mathbb{Q})$ permutes the cube roots of α and $\bar{\alpha}$.

3. Let L/K be a finite extension of fields. For $\theta \in L$, define $\mathrm{Tr}(\theta)$ to be the trace of the map $x \mapsto \theta x$, where we are thinking of this map as $[L : K] \times [L : K]$ matrix with entries in K.
   (a) Show that $\mathrm{Tr}(\theta_1 + \theta_2) = \mathrm{Tr}(\theta_1) + \mathrm{Tr}(\theta_2)$. Show that, for $a \in K$, we have $\mathrm{Tr}(a\theta) = a\mathrm{Tr}(\theta)$.
   (b) Suppose that $L = K(\theta)$, and $f(x) = x^n + f_{n-1}x^{n-1} + \cdots + f_1 x + f_0$ is the minimal polynomial of θ. Show that $\mathrm{Tr}(\theta) = -f_{n-1}$.
   (c) Let $f(x)$ be the minimal polynomial of θ and let $[L : K(\theta)] = m$. Show that $\mathrm{Tr}(\theta) = -m f_{n-1}$.
   (d) Let L/K be Galois with Galois group G. Show that $\mathrm{Tr}(\theta) = \sum_{g \in G} g\theta$.
   (e) (**Harder**) Suppose that $L = K(\theta)$ and $x^n + f_{n-1}x^{n-1} + \cdots + f_1 x + f_0$ is the minimal polynomial of θ. Define the polynomial $g(x) = 1 + f_{n-1}x + \cdots + f_0 x^n = x^n f(1/x)$. Show that:
   $$\sum_{j=1}^{\infty} \mathrm{Tr}(\theta^j) x^j = -x \frac{g'(x)}{g(x)}.$$
   Show that, if f' is not the zero polynomial, then $\mathrm{Tr}(\theta^j) \neq 0$ for some j.

4. Let L/K be a Galois extension with Galois group G. Let F be a field with $K \subset F \subset L$ and let H be the subgroup of G fixing F. Let $X \subset G$ be the set of $g \in G$ with $gF = F$.
   (a) Describe X in terms of purely group theoretic operations involving G and H.
   (b) Let $A = \mathrm{Aut}(F/K)$. Describe a map $X \to A$ and show that it is a surjection.
   (c) Describe A in terms of purely group theoretic operations involving G and H.