THE SECOND HALF OF THE SCHUR-ZASSENHAUS THEOREM

The goal of this note is to prove the following result: Let A be an abelian group and B a finite group such that multiplication by $|B|$ is invertible on A. Let $1 \to A \to G \to B \to 1$ be a short exact sequence. Then this sequence is right split, meaning there is a map of groups $B \to G$ with $B \to G \to B$ the identity.

We will write the group A additively, and G and B multiplicatively, so we write $\exp : A \to G$.

Since A is normal in G, there is a conjugation action of G on A. Since A is abelian, this action factors through the quotient B. We write $\rho : B \to \mathrm{Aut}(A)$ for this action. In equations, if $b \in B$ and $\tilde{b}$ is any lift of b to G, then we have

$$\tilde{b} \exp(a) = \exp(\rho(b) \cdot a)\tilde{b}.$$

Choose any set theoretic section $\sigma : B \to G$ and define $\psi : B \times B \to A$ by

$$\sigma(b_1)\sigma(b_2) = \exp(\psi(b_1, b_2))\sigma(b_1 b_2).$$

So $\psi = 0$ if and only if σ is a map of groups.

We note that ρ and ψ determine the multiplication in G, since we have

$$\exp(a_1)\sigma(b_1)\exp(a_2)\sigma(b_2) = \exp(a_1 + \rho(b_1) \cdot a_2)\sigma(b_1)\sigma(b_2) = \exp(a_1 + \rho(b_1) \cdot a_2 + \psi(b_1, b_2))\sigma(b_1 b_2). \tag{1}$$

We want to show that we can modify the section σ to be a map of groups. Any such modification is of the form $b \mapsto \exp(-\alpha(b))\sigma(b)$ for some map of sets $\alpha : B \to A$.

We want

$$\exp(-\alpha(b_1))\sigma(b_1)\exp(-\alpha(b_2))\sigma(b_2) = \exp(-\alpha(b_1 b_2))\sigma(b_1 b_2).$$

Using Equation (1), this is the same as

$$\exp(-\alpha(b_1) - \rho(b_1) \cdot \alpha(b_2) + \psi(b_1, b_2))\sigma(b_1 b_2) = \exp(-\alpha(b_1 b_2))\sigma(b_1 b_2)$$

or, in other words,

$$\psi(b_1, b_2) = \alpha(b_1) + \rho(b_1) \cdot \alpha(b_2) - \alpha(b_1 b_2). \tag{2}$$

We compute $\sigma(b_1)\sigma(b_2)\sigma(b_3)$ in two ways:

$$\begin{array}{rcccl} [\sigma(b_1)\sigma(b_2)]\,\sigma(b_3) &=& \exp(\psi(b_1, b_2))\sigma(b_1 b_2)\sigma(b_3) &=& \exp(\psi(b_1, b_2) + \psi(b_1 b_2, b_3)) \quad \sigma(b_1 b_2 b_3) \\ \sigma(b_1)\,[\sigma(b_2)\sigma(b_3)] &=& \sigma(b_1)\exp(\psi(b_2, b_3))\sigma(b_2 b_3) &=& \exp(\rho(b_1) \cdot \psi(b_2, b_3) + \psi(b_1, b_2 b_3))\ \sigma(b_1 b_2 b_3). \end{array}$$

Comparing the right hand sides, and taking the opportunity to rearrange,

$$\psi(b_1, b_2) = \psi(b_1, b_2 b_3) + \rho(b_1) \cdot \psi(b_2, b_3) - \psi(b_1 b_2, b_3). \tag{3}$$

So we know (3) and want to deduce (2). Since $|B|$ is invertible in A, we can average equation (3) over b_3. As b_3 ranges over B, so does $b_2 b_3$, so we deduce

$$\psi(b_1, b_2) = \frac{1}{|B|} \sum_{b \in B} \psi(b_1, b) + \rho(b_1) \cdot \frac{1}{|B|} \sum_{b \in B} \psi(b_2, b) - \frac{1}{|B|} \sum_{b \in B} \psi(b_1 b_2, b).$$

Defining

$$\alpha(b') = \frac{1}{|B|} \sum_{b \in B} \psi(b', b),$$

we see that α satisfies (2). $\square$

Remark: There are related results in the theory of Lie groups: If K is a compact Lie group, V a finite dimensional vector space and $1 \to V \to G \to K \to 1$ a short exact sequence, then this sequence is right split. We just replace the average over B by an integral over K. We can do more. Define a Lie group U to be unipotent if it has a central filtration whose quotients are vector spaces. Repeating the earlier steps in the deduction of the Schur-Zassenhaus theorem, we can deduce that any short exact sequence $1 \to U \to G \to K \to 1$ with U unipotent and K compact is right split.