We recall the terminology: Let

$$1 \to N \xrightarrow{i} G \xrightarrow{q} Q \to 1$$

be a short exact sequence of groups.

We say this sequence is **left split** and G is the **direct product of N and Q** if there is a map

$$1 \to N \underset{j}{\overset{i}{\rightleftarrows}} G \xrightarrow{q} Q \to 1$$

such that $N \to G \to N$ is the identity. In this case, $G \cong N \times Q$.

We say that the sequence is **right split** and G is the **semidirect product of N and Q** is there is a map

$$1 \to N \xrightarrow{i} G \underset{p}{\overset{q}{\rightleftarrows}} Q \to 1$$

such that $Q \to G \to Q$ is the identity. In this case, we abuse notation and write $G \cong N \rtimes Q$. More carefully, define $\phi : Q \to \mathrm{Aut}(N)$ by $\phi(g)(n) = p(g)np(g)^{-1}$. Then G, and the whole sequence above, is determined up to isomorphism by ϕ and we write $G \cong N \rtimes_\phi Q$.

Which of the following sequences are direct products, which are semidirect products, and which do not split at all?

(1) $$1 \to C_3 \to C_{15} \to C_5 \to 1$$

(2) $$1 \to C_3 \to C_9 \to C_3 \to 1$$

(3) $$1 \to A_5 \to S_5 \to C_2 \to 1$$

For a commutative ring R, $GL_n(R)$ is defined to be the $n \times n$ matrices with entries in R who determinant is a unit of R.

(4) $$1 \to \mathrm{SL}_3(\mathbb{F}_5) \to \mathrm{GL}_3(\mathbb{F}_5) \xrightarrow{\det} \mathbb{F}_5^\times \to 1$$

(5) $$1 \to \mathrm{SL}_3(\mathbb{F}_7) \to \mathrm{GL}_3(\mathbb{F}_7) \xrightarrow{\det} \mathbb{F}_7^\times \to 1$$

For a prime p:

(6) $$1 \to C_p^4 \to \mathrm{GL}_2(\mathbb{Z}/p^2\mathbb{Z}) \to \mathrm{GL}(\mathbb{F}_p) \to 1$$

For p an odd prime, the Heisenberg group H is defined to be $H = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{F}_p \right\}$.

(7) $$1 \to C_p \xrightarrow{z \mapsto \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} H \to C_p^2 \to 1$$

(8) $$1 \to C_p^2 \xrightarrow{(y,z) \mapsto \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}} H \to C_p \to 1$$