# 10. Solvable groups

We recall that the commutator subgroup $[G, G]$ of a group $G$ is the subgroup generated by all products $ghg^{-1}h^{-1}$.

**Definition.** The ***derived series*** of $G$ is the sequence of subgroups $D_0(G) \supseteq D_1(G) \supseteq D_2(G) \supseteq \cdots$ defined inductively by $D_0(G) = G$ and $D_k(G) = [D_{k-1}(G),\ D_{k-1}(G)]$. We call $D_k(G)$ the ***$k$-th derived subgroup***.

**Remark.** The $k$-derived subgroup is often also denoted $G^{(k)}$. The parentheses in $G^{(k)}$ is meant to distinguish $G^{(k)}$ from the $k$-fold product of $G$ with itself. Professor Speyer recommends, instead, using words to do this: Say "let $G^{(k)}$ be the $k$-th derived subgroup ..." or "let $G^k$ be the $k$-fold product of $G$ with itself ...."

**Problem 10.1.** Check that the derived series of $S_4$ is $S_4 \supset A_4 \supset V \supset \{e\}$ where $V$ is the four element group generated by $(12)(34)$ and $(13)(24)$.

**Definition.** A group $G$ is called ***solvable*** if there is some index $N$ such that the $N$-th derived subgroup is trivial.

**Problem 10.2.** Show that a group $G$ is solvable if and only if it has a subnormal series in which each subquotient is abelian.

**Problem 10.3.** Show that subgroups of solvable groups are solvable.

**Problem 10.4.** Show that quotients of solvable groups are solvable.

**Problem 10.5.** Let $1 \to A \to B \to C \to 1$ be a short exact sequence with $A$ and $C$ solvable. Show that $B$ is solvable.

**Problem 10.6.** Let $G$ be the group of bijections $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ of the form $x \mapsto ax + b$. Show that $G$ is solvable.

**Problem 10.7.** Let $k$ be a field and let $B$ be the group of invertible upper triangular $n \times n$ matrices with entries in $k$. Show that $B$ is solvable.