11. Center, central series and nilpotent groups

Definition. The ***center*** of a group G is the set $Z(G) := \{h : \; gh = hg \; \forall g \in G\}$.

Problem 11.1. Let G be a group.

(1) Check that $Z(G)$ is a subgroup of G.
(2) Check that $Z(G)$ is canonical in G (and hence normal).
(3) Check that every subgroup of $Z(G)$ is normal in G.

Problem 11.2. Let k be a field and let U be the group of matrices with entries in k of the form $\left[\begin{smallmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{smallmatrix}\right]$. Show that the center of U is the group of matrices of the form $\left[\begin{smallmatrix} 1 & 0 & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right]$.

Problem 11.3. Check that the center of S_n is trivial for $n \geq 3$.

Problem 11.4. Let F be a field with more than two elements. Let B be the group of matrices of the form $\left[\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right]$. Show that the center of F is $\{\left[\begin{smallmatrix} z & 0 \\ 0 & z \end{smallmatrix}\right] : z \in F^\times\}$.

This problem was on the problem sets in a slightly different form; check that everyone in your group remembers how to do it.

Problem 11.5. Let p be a prime and let G be a group of order p^k for some $k \geq 1$. Show that $Z(G)$ is nontrivial.

Definition. Let G be a group. A ***central series*** of G is a sequence of subgroups $G_0 \triangleleft G_1 \triangleleft G_2 \triangleleft \cdots \triangleleft G_N$ such that, if $g \in G$ and $h \in G_i$ then $ghg^{-1}h^{-1} \in G_{i-1}$, for $1 \leq i \leq N$. G is called ***nilpotent*** if it has a central series $G_0 \triangleleft G_1 \triangleleft G_2 \triangleleft \cdots \triangleleft G_N$ with $G_0 = \{e\}$ and $G_N = G$.

Remark. In many sources, a central series is required to have $G_0 = \{e\}$ and $G_N = G$, but then the "upper central series" and the "lower central series", which you will meet on the problem sets, are not central series. I prefer to take the more general definition.

Problem 11.6. Let $G_0 \triangleleft G_1 \triangleleft G_2 \triangleleft \cdots \triangleleft G_N$ be a series of subgroups of G. Show that G is a central series if and only if all the G_i are normal in G, and $G_i/G_{i-1} \subseteq Z(G/G_{i-1})$ for $1 \leq i \leq N$.

Problem 11.7. Let k be a field and let U be the group of matrices with entries in k of the form

$$\begin{bmatrix} 1 & * & * & \cdots & * \\ & 1 & * & \cdots & * \\ & & 1 & \cdots & * \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}.$$

Show that U is nilpotent.

Problem 11.8. Let p be a prime and let G be a group of order p^k for some $k \geq 1$. Show that G is nilpotent.

Problem 11.9. Show that a nilpotent group is solvable.

Problem 11.10. Show that a subgroup of a nilpotent group is nilpotent.

Problem 11.11. Show that a quotient of a nilpotent group is nilpotent.

Problem 11.12. Show that the following groups are solvable but not nilpotent.

(1) The symmetric groups S_3 and S_4.
(2) The group of invertible matrices of the form $\left[\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right]$ with entries in a field with more than two elements.

Problem 11.13. Give an example of a short exact sequence $1 \to A \to B \to C \to 1$ with A and C nilpotent but B not nilpotent.