12. THE SYLOW THEOREMS

Let p be a prime.

Definition. A ***p*-group** is a group P with $\#(P) = p^k$ for some k. For a group G, a ***p*-subgroup** of G is a subgroup which is a p-group.

Problem 12.1. Let P be a p group and let X be a finite set on which P acts. Suppose that $\#(X) \not\equiv 0 \bmod p$. Show that P fixes some point of X.

Let G be a group. Factor $\#(G)$ as $p^k m$ where p does not divide m.

Definition. A ***Sylow p*-subgroup** of G is a subgroup of G of order p^k.

Large parts of the following problems appeared on the homework; please remind each other of the solutions.

Problem 12.2. Let $\mathrm{GL}_n(\mathbb{F}_p)$ be the group of $n \times n$ matrices with entries in the field with p elements.

(1) Show that $\# \mathrm{GL}_n(\mathbb{F}_p) = \prod_{j=0}^{n-1} (p^n - p^j)$.
(2) Show that $\mathrm{GL}_n(\mathbb{F}_p)$ has a Sylow p-subgroup.

Problem 12.3. Let $v(n)$ be the exponent such that $n! = p^{v(n)} m$ with p not dividing m.

(1) Write $n = pm + r$ with $0 \leq r \leq p-1$. Show that $v(n) = m + v(m)$.
(2) Show that S_n has a Sylow p-subgroup.

Problem 12.4. Let Γ be a finite group with a Sylow p-subgroup Π. Let G be a subgroup of Γ.

(1) Show that G has a Sylow p-subgroup P. Hint: Consider G acting on Γ/Π.
(2) Show, more specifically, that there is some $\gamma \in \Gamma$ such that $P = G \cap \gamma \Pi \gamma^{-1}$.

Hint for the following three problems: Use Problem 12.4.

Problem 12.5. (**The first Sylow theorem**) Show that every finite group G has a Sylow p-subgroup.

Problem 12.6. Let G be a finite group and let P be a Sylow p-subgroup with $\#(P) = p^k$.

(1) Let Q be a p-subgroup of G. Show that there is some $g \in G$ such that $Q \subseteq gPg^{-1}$.
(2) Let H be a subgroup of G whose order is divisible by p^k. Show that there is some $g \in G$ such that $H \supseteq gPg^{-1}$.

Problem 12.7. (**The second Sylow theorem**) Let G be a finite group and let P_1 and P_2 be two Sylow p-subgroup of G. Show that there is some $g \in G$ such that $P_2 = gP_1g^{-1}$.

Let G be a group and let H be a subgroup of G. We define $N_G(H) = \{g \in G : gHg^{-1} = H\}$. The group $N_G(H)$ is called the ***normalizer*** of H in G.

Problem 12.8. Map $G/N_G(P)$ to the set of Sylow p-subgroups by sending the coset $gN_G(P)$ to gPg^{-1}. Show that this map is well defined, and is a bijection.

Problem 12.9. (1) Show that P is normal in $N_G(P)$.
(2) Let Q be a p-subgroup of $N_G(P)$. Show that $Q \subseteq P$.
(3) Let H be a p-subgroup of G. Show that $H \cap N_G(P) = H \cap P$.

Problem 12.10. Since P is a subgroup of G, the group P acts on $G/N_G(P)$. Show that the only coset which is fixed for this action is $eN_G(P)$.

Problem 12.11. (**The third Sylow theorem**) The number of Sylow p-subgroups of G is $\equiv 1 \bmod p$.