13. Some problems with Sylow groups

Problem 13.1. Let G be a group of order $p^k m$ where p does not divide m. Show that the number of p-Sylow subgroups of G divides m.

Problem 13.2. Let G and H be finite groups and p a prime number. Let P and Q be p-Sylow subgroups of G and H.

(1) Show that $P \times Q$ is a p-Sylow subgroup of $G \times H$.
(2) Show that every p-Sylow subgroup of $G \times H$ is of the form $P' \times Q'$ for P' and Q' p-Sylow subgroups of G and H.

Problem 13.3. Let $1 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 1$ be a short exact sequence of finite groups and let Q be a p-Sylow subgroup of B. Show that $\alpha^{-1}(Q)$ and $\beta(Q)$ are p-Sylow subgroups of A and C respectively.

Problem 13.4. Let $p < q$ be primes and let G be a group of order pq.

(1) Show that the q-Sylow subgroup of G is normal.
(2) Conclude that there is a short exact sequence $1 \to C_q \to G \to C_p \to 1$.
(3) Show that $G \cong C_q \rtimes C_p$ for some action of C_p on C_q.

Problem 13.5. Show that there are no simple groups of order 40. (Hint: Look at 5-Sylows.)

Problem 13.6. In this problem, we will show that there is no simple group G of order 80.

(1) Show that, if G were such a group, then G would have five 2-Sylow subgroups.
(2) Consider the map $G \to S_5$ to get a contradiction.

Problem 13.7. Recall that, for a subgroup H of a group G, the normalizer $N_G(H)$ is defined to be $\{g \in G : gHg^{-1} = H\}$. Let G be a finite group and P a p-Sylow subgroup of G.

(1) Show that P is canonical in $N_G(P)$.
(2) Show that $N_G(N_G(P)) = N_G(P)$.

Problem 13.8. Let G be a finite nilpotent group. On the homework you showed/will show that, if $H \subsetneq G$ is a proper subgroup, then $N_G(H) \supsetneq H$.

(1) Show that every Sylow p-subgroup of a finite nilpotent group G is normal.
(2) Let P and Q be Sylow subgroups of G for different primes, p and q. Show that, if $g \in P$ and $h \in Q$, then $gh = hg$.
(3) Let G_p be the Sylow p-subgroup of G. Show that $G \cong \prod G_p$, where the right hand side is the direct product.

In other words, every finite nilpotent group is the direct product of its Sylow subgroups.

Problem 13.9. Let $1 \to A \to B \xrightarrow{\beta} C \to 1$ and let P be a Sylow p-subgroup of A. We'll identify A with its image in B.

(1) (**Frattini's argument**) Show that $B = AN_B(P)$. Hint: Let $b \in B$. What can you say about nPb^{-1}?
(2) Show that $N_B(P) \xrightarrow{\beta} C$ is surjective.
(3) Show that $1 \to N_A(P) \to N_B(P) \to C \to 1$ is exact.