14. SCHUR-ZASSENHAUS, THE ABELIAN CASE

The aim of the next two worksheets will be to prove:

Theorem (Schur-Zassenhaus)**.** Let $1 \to A \to B \to C \to 1$ be a short exact sequence of finite groups where $\mathrm{GCD}(\#(A), \#(C)) = 1$. Then this sequence is right split, so $B \cong A \rtimes C$.

This is the start of an answer to the question "how are groups assembled out of smaller groups": When you put groups of relatively prime order together, you just get semidirect products.

Today, we'll be proving the case where A is abelian.[1] Here is our main result:

Today's goal: Let A be an abelian group, C a finite group of size n, and suppose that $a \mapsto a^n$ is a bijection from A to A. Let $1 \to A \to B \to C \to 1$ be a short exact sequence. Then this sequence is right split.

Problem 14.1. Show that, if A is a finite abelian group and n an integer such that $\mathrm{GCD}(\#(A), n) = 1$, then $a \mapsto a^n$ is a bijection. Thus, the above Theorem does imply the Schur-Zassenhaus theorem for A abelian.

From now on, let A be an abelian group, let C be a finite group and let $1 \to A \to B \xrightarrow{\beta} C \to 1$ be a short exact sequence. We abbreviate $\#(C)$ to n; we will not introduce the hypothesis on $a \mapsto a^n$ until later. We'll identify A with its image in B.

Let $\mathcal{S}$ be the set of right inverses of β, meaning maps $\sigma : C \to B$ such that $\beta(\sigma(c)) = c$. We emphasize that σ is not required to be compatible with the group multiplication in any way. Let B act on $\mathcal{S}$ by $(b\sigma)(c) = b\sigma(\beta(b)^{-1}c)$.

Problem 14.2. Check that this is an action.

Let σ_1 and $\sigma_2 \in \mathcal{S}$. Set

$$d(\sigma_1, \sigma_2) = \prod_{c \in C} \left(\sigma_1(c)\sigma_2(c)^{-1}\right). \qquad (*)$$

We don't have to specify the order of the product, because every term is in A.

Problem 14.3. Show that $d(\sigma_1, \sigma_2)d(\sigma_2, \sigma_3) = d(\sigma_1, \sigma_3)$ and $d(\sigma_1, \sigma_2) = d(\sigma_2, \sigma_1)^{-1}$.

Problem 14.4. For the action of B on $\mathcal{S}$ described above, check that $d(b\sigma_1, b\sigma_2) = bd(\sigma_1, \sigma_2)b^{-1}$.

Define $\sigma_1 \equiv \sigma_2$ if $d(\sigma_1, \sigma_2) = 1$.

Problem 14.5. Check that $\equiv$ is an equivalence relation.

Define $\mathcal{X}$ to be the set of equivalence classes of $\mathcal{S}$ module the relation $\equiv$.

Problem 14.6. Check that the action of B on $\mathcal{S}$ descends to an action of B on $\mathcal{X}$.

Now, we impose the condition that $a \mapsto a^n$ is an automorphism of A.

Problem 14.7. Show that the subgroup A of B acts on $\mathcal{X}$ with a single orbit and trivial stabilizers.

The following problem was on the problem sets; check that everyone knows how to do it:

Problem 14.8. You have now shown that B acts on $\mathcal{X}$, and that the restriction of this action to A has a single orbit and trivial stabilizers. Explain why this means that $1 \to A \to B \to C \to 1$ is right split.

Remark. For this remark, I'll switch to writing A additively. There are useful situations where A is infinite but we can still show $a \mapsto na$ is bijective. For example, we can consider $0 \to V \to B \to C \to 1$ where C is finite and V is a vector space over a field of characteristic zero. A more sophisticated examples is short exact sequences of Lie groups $0 \to \mathbb{R}^k \to G \to K \to 1$ where K is compact; here we replace the product in $(*)$ with $\int_{k \in K} \sigma_1(k)\sigma_2^{-1}(k)$.

[1] This approach is closely based on that of Kurzweil and Stellmacher, *The Theory of Finite Groups*, Chapter 3.3, Springer-Verlag (2004).