16. Simplicity of $\mathrm{PSL}_n(F)$

The three most important families of simple groups are

- The cyclic groups C_p for p prime.
- The alternating groups A_n for $n \geq 5$.
- The projective special linear groups $\mathrm{PSL}_n(F)$, except for $\mathrm{PSL}_2(\mathbb{F}_2)$ and $\mathrm{PSL}_2(\mathbb{F}_3)$.

In this worksheet, we'll show $\mathrm{PSL}_n(F)$ is simple for $\#(F) > 5$. The fields of orders 2, 3, 4 and 5 are not deeper, but the details are messier.

Let F be a field. The groups $\mathrm{GL}_n(F)$ and $\mathrm{SL}_n(F)$ are the groups of $n \times n$ matrices with entries in F which, respectively, have nonzero determinant and have determinant 1. Let Z be the group of matrices of the form $z\,\mathrm{Id}_n$, for $z \in F^{\times}$. The ***projective general linear group*** and ***projective special linear group*** are, respectively, $\mathrm{PGL}_n(F) := \mathrm{GL}_n(F)/Z$ and $\mathrm{PSL}_n(F) := \mathrm{SL}_n(F)/(Z \cap \mathrm{SL}_n(F))$.

For $1 \leq i \neq j \leq n$ and $r \in F$, the matrix $E_{ij}(r)$ is the $n \times n$ matrix with ones on the diagonal, an r in position (i,j) and zeroes everywhere else. A matrix of the form $E_{ij}(r)$ is called an ***elementary matrix***. We proved in 593 (and you may use) that the elementary matrices generate $\mathrm{SL}_n(F)$ for any F.

Problem 16.1. Let N be a normal subgroup of $\mathrm{SL}_n(F)$. Suppose that there a pair of indices (a,b) so that N contains all the matrices $E_{ab}(r)$. Show that $N = \mathrm{SL}_n(F)$.

A ***companion matrix*** is a matrix of the form

$$\begin{bmatrix} 0 & 0 & 0 & \cdots & * \\ 1 & 0 & 0 & \cdots & * \\ 0 & 1 & 0 & \cdots & * \\ & & & & \\ 0 & 0 & \ddots & \cdots & * \\ 0 & 0 & \cdots & 1 & * \end{bmatrix}.$$

By the Rational Canonical Form Theorem, for $\alpha \in \mathrm{GL}_n(F)$, there is $g \in \mathrm{GL}_n(F)$ such that $g\alpha g^{-1}$ is block diagonal with blocks that are companion matrices and, furthermore, we can take the largest block to have size equal to the degree of the minimal polynomial of α. We will want a variant of this for $\mathrm{SL}_n(F)$.

Problem 16.2. Define a ***generalized companion matrix*** to be an $m \times m$ matrix of the form

$$\begin{bmatrix} 0 & 0 & 0 & \cdots & * \\ * & 0 & 0 & \cdots & * \\ 0 & * & 0 & \cdots & * \\ & & & & \\ 0 & 0 & \ddots & \cdots & * \\ 0 & 0 & \cdots & * & * \end{bmatrix}.$$

(1) Let $\alpha \in \mathrm{SL}_n(F)$. Show that there is $h \in \mathrm{SL}_n(F)$ such that $h\alpha h^{-1}$ is block diagonal with blocks that are generalized companion matrices.
(2) If $\alpha \notin Z$, show furthermore that we can assume the largest block has size ≥ 2.

Problem 16.3. Let β be an $m \times m$ generalized companion matrix for $m \geq 2$. Assume that $\#(F) > 5$. Show that we can find a diagonal matrix $d \in \mathrm{SL}_n(F)$ such that $d^{-1}\beta^{-1}d\beta$ is of the form

$$\begin{bmatrix} \gamma_1 & 0 & 0 & \cdots & * \\ 0 & \gamma_2 & 0 & \cdots & * \\ 0 & 0 & \gamma_3 & \cdots & * \\ & & & & \\ 0 & 0 & 0 & \ddots & * \\ 0 & 0 & 0 & \cdots & \gamma_m \end{bmatrix} \qquad (*)$$

with $\gamma_1 \neq \gamma_m$. Hint: I found it helpful to write

$$\begin{bmatrix} 0 & 0 & 0 & \cdots & * \\ * & 0 & 0 & \cdots & * \\ 0 & * & 0 & \cdots & * \\ & & & & \\ 0 & 0 & \ddots & \cdots & * \\ 0 & 0 & \cdots & * & * \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ & & & & \\ 0 & 0 & \ddots & \cdots & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \begin{bmatrix} * & 0 & 0 & \cdots & * \\ 0 & * & 0 & \cdots & * \\ & & & & \\ 0 & 0 & \ddots & \cdots & * \\ 0 & 0 & \cdots & * & * \\ 0 & 0 & 0 & \cdots & * \end{bmatrix}.$$

Problem 16.4. Let $m \geq 2$ and let N be a normal subgroup of $\mathrm{SL}_m(F)$ containing a matrix γ of the form $(*)$ with $\gamma_1 \neq \gamma_m$. Show that N contains all matrices of the form $E_{1m}(r)$. Hint: Compute $\gamma^{-1}E_{1m}(s)^{-1}\gamma E_{1m}(s)$.

Problem 16.5. Let $\#(F) > 5$ and let N be a normal subgroup of $\mathrm{SL}_n(F)$ not contained in Z. Show $N = \mathrm{SL}_n(F)$.

Problem 16.6. Let $\#(F) > 5$. Show that $\mathrm{PSL}_n(F)$ is simple.

Remark. The assumption that $\#(F) > 5$ was used in Problem 16.3. A case by case analysis can derive the same conclusion for $\#(F) = 4, 5$ and for $\#(F) = 3$ with $n \geq 3$. If $\#(F) = 2$, it is impossible to have $\gamma_1 \neq \gamma_m$, but a case by case analysis can show that, for $n \geq 3$ a normal subgroup of $\mathrm{SL}_n(\mathbb{F}_2)$ containing a generalized companion matrix with a block of size ≥ 2 contains an elementary matrix, and then the proof finishes as before.

Remark. The slickest proof that $\mathrm{PSL}_n(F)$ is simple uses Iwasawa's Criterion, but that is not closely related to other material we have covered. See Keith Conrad's lecture notes at `https://kconrad.math.uconn.edu/blurbs/grouptheory/PSLnsimple.pdf` for a good exposition of that approach.

Remark. $\mathrm{PSL}_n(F)$ is an example of a "group of Lie type", which roughly means to take a complex simple Lie group like $\mathrm{PSL}_n(\mathbb{C})$ and make "the same definition over a general field". The complex simple Lie groups are classified. A given complex simple Lie group can correspond to more than one group of Lie type, but the groups of Lie type are also classified. The Classification of Finite Simple Groups says that every finite simple group is either cyclic, alternating, of Lie type, or in a list of 26 sporadic examples. The status of the CFSG is a little unclear; a proof was announced in 1983, with the argument spread over hundreds of papers occupying tens of thousands of pages. Two gaps in the argument were found, and fixed in 2004 and 2008 respectively, and no new ones have been found since then. Group theorists are currently at work to produce a shorter, cohesive proof.