17. Review of polynomial rings

Throughout this worksheet, let k be a field. We write $k[x]$ for the ring of polynomials with coefficients in $k[x]$.

Problem 17.1. Let $b(x) \in k[x]$ be a nonzero polynomial of degree d. Let $a(x)$ be any polynomial in $k[x]$. Show that there are unique polynomials $q(x)$ and $r(x)$, with $\deg r < d$, such that

$$a(x) = b(x)q(x) + r(x).$$

In other words, $k[x]$ is a Euclidean ring with respect to the norm of degree. As you hopefully remember, this means that $k[x]$ is a PID and a UFD, and we can compute GCD's using the Euclidean algorithm.

Problem 17.2. Let $b(x) \in k[x]$ be a nonzero polynomial of degree d. Show that the ring $k[x]/b(x)k[x]$ is a k-vector space of dimension d.

Problem 17.3. Use the Euclidean algorithm to compute the GCD of $t^3 + t$ and $t^4 - 1$ in $\mathbb{Q}[t]$

Problem 17.4. Use the Euclidean algorithm to find polynomials $f(t)$ and $g(t) \in \mathbb{Q}[t]$ such that

$$f(t)(t^2 + t + 2) + g(t)(t^3 - 2) = 1.$$

Problem 17.5. Let $b(x)$ be an irreducible polynomial in $k[x]$. Show that $k[x]/b(x)k[x]$ is a field.

Problem 17.6. The polynomial $t^3 - 2$ is irreducible in $\mathbb{Q}[t]$, so the previous problem says that $\mathbb{Q}[t]/(t^3 - 2)\mathbb{Q}[t]$ is a field. Compute $(t^2 + t + 2)^{-1}$ in this field.

Let K be a larger field containing k. For $\theta \in K$, we say that θ is ***algebraic*** over k if there is a nonzero polynomial $f(t)$ in $k[t]$ with $f(\theta) = 0$.

Problem 17.7. Let $\theta \in K$ be algebraic over k. Let $I \subset k[t]$ be $\{f(\theta) \in k[t] : f(\theta) = 0\}$.

(1) Show that I is an ideal.
(2) Show that $I = m(t)k[t]$ for some irreducible polynomial m.
(3) Show that $k[\theta]$, meaning the subring of K generated by k and θ, is isomorphic to $k[t]/m(t)k[t]$.

The polynomial $m(t)$ is called the ***minimal polynomial*** of θ.

Problem 17.8. Show that θ is algebraic over k if and only if $\dim_k k[\theta] < \infty$.

Problem 17.9. Show that the set of elements of K which are algebraic over k is a subfield of K.