18. DEGREES OF FIELD EXTENSIONS, AND CONSTRUCTIBLE NUMBERS

Definition: Let L be a field and K a subfield. The ***degree of*** L ***over*** K, written $[L : K]$, is the dimension of L as a K-vector space.

Problem 18.1. Let $K \subseteq L \subseteq M$ be three fields with $[L : K]$ and $[M : L] < \infty$. Show that $[M : K] = [M : L][L : K]$.

Problem 18.2. Let $k \subseteq K$ be a field extension with $[K : k] < \infty$. Let $\theta \in K$ and let $m(x)$ be the minimal polynomial of θ over k. Show that $\deg m(x)$ divides $[K : k]$.

We illustrate these results with an extremely classical application. A real number $\theta \in \mathbb{R}$ is called ***constructible*** if it can be written in terms of rational numbers using the operations $+$, $-$, $\times$, $\div$ and $\sqrt{\ }$. Classically, these numbers were studied because the distance between any two points constructed with straightedge and compass is constructible; now we can motivate them by saying they are the numbers which can be computed exactly with a four function calculator.

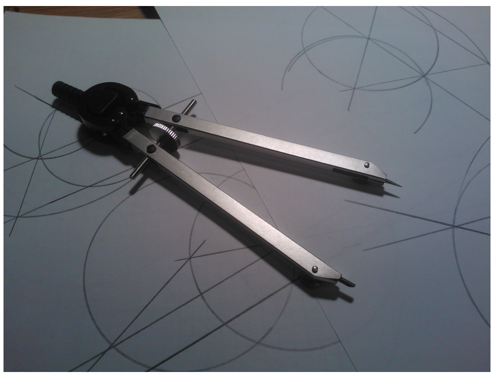

Figure: Two ancient mathematical tools

Problem 18.3. Suppose we compute a sequence of real numbers $\theta_1, \theta_2, \theta_3, \ldots, \theta_N$ where each θ_k is either

- a rational number,
- of one of the forms $\theta_i + \theta_j$, $\theta_i - \theta_j$, $\theta_i\theta_j$ or θ_i/θ_j for some $i, j < k$ or
- of the form $\sqrt{\theta_j}$ for some $j < k$.

Show that $[\mathbb{Q}[\theta_1, \theta_2, \ldots, \theta_N] : \mathbb{Q}]$ is a power of 2.

Problem 18.4. Let θ be a constructible real number and let $m(x)$ be its minimal polynomial over $\mathbb{Q}$. Show that $\deg m(x)$ is a power of 2.

Problem 18.5. (**The impossibility of doubling the cube**.) Show that $\sqrt[3]{2}$ is not constructible.

Problem 18.6. (**The impossibility of trisecting the angle**) It is well known that a $60°$ angle is constructible with straightedge and compass. Show, however, that $\cos 20°$ is not constructible. Hint:

$$4\cos^3 20° - 3\cos 20° = \cos 60° = \frac{1}{2}.$$