# PROBLEM SET 1: DUE WEDNESDAY, JANUARY 15

Please see the course website for guidance on collaboration and formatting your problem sets.

**Problem 1.1.** Let $C$ be the cube $[-1,1]^3$ in $\mathbb{R}^3$ and let $G$ be the group of rotation and reflection symmetries of $C$. You needn't prove your answers to this problem correct.

(1) Describe the $G$-orbits of the points $(1,0,0)$, $(1,1,0)$, $(1,1,1)$.
(2) For each of the points $x$ in the previous part, compute $\#\mathrm{Stab}(x)$.

**Problem 1.2.** Let $S_4$ act on the polynomial ring $\mathbb{C}[r_1,r_2,r_3,r_4]$ by permuting the variables. For each of the following polynomials, compute the size of its stabilizer. You needn't prove your answers to this problem correct.

$$f = r_1 \qquad g = r_1r_2 + r_3r_4 \qquad h = \prod_{1\le i<j\le 4}(r_i - r_j)$$

**Problem 1.3.** Let $G$ be a group and let $a$ be an element of $G$. Show that $g \mapsto aga^{-1}$ is a group homorphism $G \to G$.

**Problem 1.4.** Let $G$ and $H$ be two groups and let $\alpha : G \to H$ and $\beta : G \to H$ be group homomorphisms. For $g \in G$, define $(\alpha\beta)(g) = \alpha(g) * \beta(g)$.

(1) If $H$ is commutative, show that $\alpha\beta$ is a group homomorphism.
(2) Give an example of two groups and two homomorphisms such that $\alpha\beta$ is not a homomorphism.

**Problem 1.5.** Let $S_n$ act on $\mathbb{Q}[x_1,x_2,\ldots,x_n]$ in the obvious way. Let $\Delta = \prod_{1\le i<j\le n}(x_i - x_j) \in \mathbb{Q}[x_1,x_2,\ldots,x_n]$. For $\sigma$ in $S_n$, let $M(\sigma)$ be the corresponding permutation matrix in $\mathrm{GL}_n(\mathbb{Q})$.

(1) Show that
$$\frac{\sigma(\Delta)}{\Delta} = (-1)^{\left\{\#\{(i,j):\ 1\le i<j\le n,\ \sigma(i)>\sigma(j)\}\right\}} = \det M(\sigma)$$
(2) We define $\epsilon(\sigma)$ to be given by any of the above formulas. Show that $\epsilon$ is a group homomorphism $S_n \to \{\pm 1\}$.

The ***alternating group*** $A_n$ is defined as the kernel of $\epsilon$.

(3) Show that $S_n$ is generated by the elements $(1j)$ for $2 \le j \le n$.
(4) Show that $A_n$ is generated by the elements $(12k)$ for $3 \le k \le n$.

**Problem 1.6.** In this problem, we will prove the ***fundamental theorem of symmetric functions***. Let $k$ be a field, let $R$ be the ring $k[r_1,r_2,\ldots,r_n]$ and let $S$ be the ring of $S_n$ invariant polynomials in $R$. For $1 \le d \le n$, define

$$e_d = \sum_{1\le i_1<i_2<\cdots<i_d\le n} r_{i_1}r_{i_2}\cdots r_{i_d}.$$

The fundamental theorem of symmetric functions states that $S$ is generated (as a $k$-algebra) by $e_1, e_2, \ldots, e_n$. We'll write $E$ for the sub-$k$-algebra of $S$ generated by $e_1, e_2, \ldots, e_n$, so our goal is to show that $E = S$.

(1) Show that it is enough to show that, if $f \in S$ is homogenous of degree $m$, then $f \in E$.

Let $M$ be the set of monomials $r_1^{a_1}r_2^{a_2}\cdots r_n^{a_n}$ of degree $m$. Place a total order on $M$ by saying that $r_1^{a_1}r_2^{a_2}\cdots r_n^{a_n} > r_1^{b_1}r_2^{b_2}\cdots r_n^{b_n}$ if $a_1 = b_1$, $a_2 = b_2$, …, $a_{j-1} = b_{j-1}$ and $a_j > b_j$ for some $1 \le j \le n$. For example, if $n = 3$ and $m = 2$, our order is

$$r_1^2 > r_1r_2 > r_1r_3 > r_2^2 > r_2r_3 > r_3^2.$$

We define the ***leading term*** of a nonzero homogenous degree $m$ polynomial $f$ to be the largest monomial of $M$ whose coefficient is nonzero.

(2) Let $f$ be a nonzero homogenous degree $m$ polynomial in $S$ and let $r_1^{a_1}r_2^{a_2}\cdots r_n^{a_n}$ be the leading term of $f$. Show that $a_1 \ge a_2 \ge \cdots \ge a_n$.
(3) Let $a_1 \ge a_2 \ge \cdots \ge a_n$ with $\sum a_j = m$. Show that there is a polynomial $g \in E$ with leading term $r_1^{a_1}r_2^{a_2}\cdots r_n^{a_n}$.
(4) Prove that $S = E$.