1. The Quadratic, Cubic and Quartic formulas

> *I added up the area of my two squares:* 1300. *The side of one exceeds the side of the other by* 10.
>
> Babylonian tablet, 2000-1600 BCE, British Museum

Problem 1.1. Let $x^2 + bx + c$ be a polynomial with complex coefficients and let its roots be α_1 and α_2. Express the following quantities in terms of α_1 and α_2. In the expressions with a square root, you may choose which square root to use.

$$b \qquad c \qquad b^2 - 4c \qquad \sqrt{b^2-4c} \qquad \frac{-b+\sqrt{b^2-4c}}{2}.$$

Problem 1.2. Let the symmetric group S_2 act by switching α_1 and α_2. Describe the effect of S_2 on each of the expressions you derived in Problem 1.1.

> *When the cube with the cose beside it / equates itself to some other whole number ...* Tartaglia, 1543

Let $\omega = \frac{-1+\sqrt{-3}}{2}$; we recall that $\omega^2 + \omega + 1 = 0$ and $\omega^3 = 1$. Let β_1, β_2 and β_3 be complex numbers. Define:

$$\begin{array}{rcrcrcr} s &=& \beta_1 &+& \beta_2 &+& \beta_3 \\ \sigma_1 &=& \beta_1 &+& \omega\beta_2 &+& \omega^2\beta_3 \\ \sigma_2 &=& \beta_1 &+& \omega^2\beta_2 &+& \omega\beta_3 \end{array}$$

Problem 1.3. Let S_3 permute β_1, β_2, β_3.

(1) Describe how S_3 acts on $\{\sigma_1,\ \omega\sigma_1,\ \omega^2\sigma_1,\ \sigma_2,\ \omega\sigma_2,\ \omega^2\sigma_2\}$.
(2) Describe how S_3 acts on $\{\sigma_1^3,\ \sigma_2^3\}$.
(3) Show that S_3 fixes s and the coefficients of the quadratic polynomial $y^2 - f_1 y + f_2 := (y-\sigma_1^3)(y-\sigma_2^3)$.

Let $(x-\beta_1)(x-\beta_2)(x-\beta_3) = x^3 - e_1x^2 + e_2x - e_3$. To make your lives easier, here are some useful formulas:

$$\begin{aligned}
&e_1 = \beta_1+\beta_2+\beta_3 \\
&e_2 = \beta_1\beta_2+\beta_1\beta_3+\beta_2\beta_3 \qquad e_1^2 = \beta_1^2+2\beta_1\beta_2+2\beta_1\beta_3+\beta_2^2+2\beta_2\beta_3+\beta_3^2 \\
&e_3 = \beta_1\beta_2\beta_3 \qquad e_1e_2 = \beta_1^2\beta_2+\beta_1^2\beta_3+\beta_1\beta_2^2+6\beta_1\beta_2\beta_3+\beta_1\beta_3^2+\beta_2^2\beta_3+\beta_2\beta_3^2 \\
&e_1^3 = \beta_1^3+3\beta_1^2\beta_2+3\beta_1^2\beta_3+3\beta_1\beta_2^2+6\beta_1\beta_2\beta_3+3\beta_1\beta_3^2+\beta_2^3+3\beta_2^2\beta_3+3\beta_2\beta_3^2+\beta_3^3
\end{aligned}$$

$$\begin{aligned}
&\sigma_1\sigma_2 = \beta_1^2-\beta_1\beta_2-\beta_1\beta_3+\beta_2^2-\beta_2\beta_3+\beta_3^2 \\
&\sigma_1^3+\sigma_2^3 = 2\beta_1^3-3\beta_1^2\beta_2-3\beta_1^2\beta_3-3\beta_1\beta_2^2+12\beta_1\beta_2\beta_3-3\beta_1\beta_3^2+2\beta_2^3-3\beta_2^2\beta_3-3\beta_2\beta_3^2+2\beta_3^3
\end{aligned}$$

Problem 1.4. Give formulas for the following, as polynomials in e_1, e_2, e_3:

$$s \qquad \sigma_1\sigma_2 \qquad f_1 \qquad f_2.$$

Problem 1.5. Show that σ_1 and σ_2 can be computed from e_1, e_2, e_3 using the operations $+$, $-$, $\times$, $\sqrt{\ }$ and $\sqrt[3]{\ }$, together with multiplication by rational numbers and the number ω. Show how to likewise compute β_1, β_2 and β_3.

> *Given an equation in which the unknown quantity has four dimensions ... reduce it to another of the third degree, in the following manner ...*
>
> Descartes, *La Géometrié*, 1637

Let γ_1, γ_2, γ_3 and γ_4 be complex numbers. Set

$$\begin{array}{rcrcrcrcr} t &=& \gamma_1 &+& \gamma_2 &+& \gamma_3 &+& \gamma_4 \\ \tau_1 &=& \gamma_1 &+& \gamma_2 &-& \gamma_3 &-& \gamma_4 \\ \tau_2 &=& \gamma_1 &-& \gamma_2 &+& \gamma_3 &-& \gamma_4 \\ \tau_3 &=& \gamma_1 &-& \gamma_2 &-& \gamma_3 &+& \gamma_4 \end{array}$$

Problem 1.6. Let S_4 permute γ_1, γ_2, γ_3, γ_4. Describe how S_4 acts on

(1) $\{\pm\tau_1, \pm\tau_2, \pm\tau_3\}$
(2) $\{\tau_1^2, \tau_2^2, \tau_3^2\}$
(3) Show that S_4 fixes t and the coefficients of the polynomial $(x-\tau_1^2)(x-\tau_2^2)(x-\tau_3^2)$.

Problem 1.7. How would you compute the γ_i from the coefficients of the quartic $\prod(x-\gamma_i)$, using the operations $+$, $-$, $\times$, $\div$ and $\sqrt[n]{\ }$?