Definition: Let k be a field, let $f(x)$ be a polynomial in $k[x]$ and let K be an extension field of f. We will say that f ***splits in*** K if f factors as a product of linear polynomials in $K[x]$. We say that K is a ***splitting field of*** f if f splits as a product $c\prod(x-\theta_j)$ in $K[x]$ and the field K is generated by k and by the θ_j.

For example, if $k=\mathbb{Q}$ and $\theta_1, \theta_2, \ldots, \theta_n$ are the roots of $f(x)$ in $\mathbb{C}$, then $\mathbb{Q}[\theta_1, \ldots, \theta_n]$ is a splitting field of $f(x)$.

Problem 19.1. Let k be a field and let $f(x)$ be a polynomial in $k[x]$. Show that f has a splitting field. (Please do not use that every field has an algebraic closure. That is a much[1] harder result than this one.)

Problem 19.2. Let

$$f(x)=\left(x-\cos\tfrac{2\pi}{7}\right)\left(x-\cos\tfrac{4\pi}{7}\right)\left(x-\cos\tfrac{8\pi}{7}\right)=\tfrac{1}{8}\left(8x^3+4x^2-4x-1\right).$$

I promise, and you may trust me, that $f(x)$ is irreducible.[2] Let $K=\mathbb{Q}(\cos\frac{2\pi}{7})$.

(1) Show that $[K:\mathbb{Q}]=3$.
(2) Show that $f(x)$ splits in K. Hint: Use the double angle formula.
(3) Show that there is an automorphism $\sigma: K\to K$ with $\sigma(\cos\frac{2\pi}{7})=\cos\frac{4\pi}{7}$.

Problem 19.3. Let L be a splitting field for x^3-2 over $\mathbb{Q}$. Show that $[L:\mathbb{Q}]=6$. (Hint: At one point, it will be very useful to use the fact that $\mathbb{Q}[\sqrt[3]{2}]$ is a subfield of $\mathbb{R}$.)

This is a good time to discuss separable polynomials.

Definition: Let k be a field and let $f(x)$ be a polynomial in $k[x]$. We say f is ***separable*** if $\mathrm{GCD}(f(x), f'(x))=1$.

Problem 19.4. Let k be a field, let $f(x)$ be a polynomial in $k[x]$ and let K be a field where f splits as $c\prod_{j=1}^{n}(x-\theta_j)$. Show that f is separable if and only if $\theta_1, \theta_2, \ldots, \theta_n$ are distinct.

Problem 19.5. Let k be a field of characteristic zero.

(1) Show that a polynomial in $k[x]$ is separable if and only if it is square free.
(2) Show that irreducible polynomials in $k[x]$ are separable.

[1] In particular, the fact that every field embeds in an algebraically closed field uses the Axiom of Choice, and this problem does not.

[2] The most straightforward way to check this is to use the rational root theorem. The slickest is to note that $f(x+1)=\frac{1}{8}(8x^3+28x^2+28x+7)$ and apply Eisenstein's irreducibility theorem.