21. Introduction to field automorphisms

Definition: Let $K \subseteq L$ be fields. An ***automorphism*** of L is a bijection $\sigma : L \to L$ with $\sigma(x+y) = \sigma(x)+\sigma(y)$ and $\sigma(xy) = \sigma(x)\sigma(y)$. An ***automorphism of*** L ***fixing*** K is an automorphism of L obeying $\sigma(a) = a$ for all $a \in K$. We write $\mathrm{Aut}(L)$ for the automorphisms of L and $\mathrm{Aut}(L/K)$ for the automorphisms of L fixing K.

Problem 21.1. Let $K \subseteq L$ be fields. Let $f(x)$ be a polynomial in $K[x]$; let $\{\theta_1, \theta_2, \dots, \theta_r\}$ be the roots of f in L.

(1) Show that $\mathrm{Aut}(L/K)$ maps $\{\theta_1, \theta_2, \dots, \theta_r\}$ to itself.
(2) Show that stabilizer of θ_j in $\mathrm{Aut}(L/K)$ is $\mathrm{Aut}(L/K(\theta_j))$.

Problem 21.2. . Let K be a field, let f be a separable polynomial in $K[x]$, let L be a splitting field for f and let $\{\theta_1, \theta_2, \dots, \theta_n\}$ be the roots of f in L. Show that the action of $\mathrm{Aut}(L/K)$ on $\{\theta_1, \theta_2, \dots, \theta_n\}$ gives an **injection** $\mathrm{Aut}(L/K) \hookrightarrow S_n$.

Problem 21.3. Let K, f, L and $\{\theta_1, \theta_2, \dots, \theta_n\}$ be as in Problem 21.2. Let $g(x)$ be an irreducible factor of $f(x)$ in $K[x]$ and renumber the θ's so that $\{\theta_1, \theta_2, \dots, \theta_m\}$ are the roots of g in L. Show that $\{\theta_1, \theta_2, \dots, \theta_m\}$ is the $\mathrm{Aut}(L/K)$-orbit of θ_1 in L. Hint: Apply Problem 20.4 to the diagram

$$\begin{array}{ccccc} K[\theta_1] & \xleftarrow{\cong} & K[x]/g(x)K[x] & \xrightarrow{\cong} & K[\theta_j] \\ \downarrow & & & & \downarrow \\ L & \dashrightarrow & & & L \end{array}$$

Problem 21.4. Let L be the splitting field of $x^3 - 2$ over $\mathbb{Q}$. Show that $\mathrm{Aut}(L/\mathbb{Q}) \cong S_3$.

Problem 21.5. Let $L = \mathbb{Q}(\cos \frac{2\pi}{7})$. Show that $\mathrm{Aut}(L/\mathbb{Q}) \cong C_3$.