Remark: This worksheet covers a number of topics which are often glossed over in first courses on Galois theory. We could afford to gloss over them too, but this seems like the natural spot for them.

Definition Let $K \subseteq L$ be an extension of fields. An element $\theta \in L$ is called ***separable*** over K if θ is algebraic over K and the minimal polynomial of θ over K is a separable polynomial. The extension L/K is called ***separable*** if it is generated by separable elements.

Problem 23.1. Show that, if K has characteristic zero then every finite degree extension of K is separable.

So, in characteristic zero, the "separable" condition usually comes for free. At the end of the worksheet, we'll return to think harder about separability in characteristic p.

Problem 23.2. Let L/K be a separable field extension; specifically, let $\theta_1, \ldots, \theta_N$ be separable elements generating L/K and let $g_j(x)$ be the (separable) minimal polynomial of θ_j over K. Let M be the splitting field of $\prod_j g_j(x)$ over K. Show that M/K is Galois.

The field M is what we will eventually call ***the Galois closure of*** L/K, but we haven't proved any uniqueness properties of it yet. Before we address that, some more basic things.

Problem 23.3. Let K, L and M be as in the previous problem. Show that every element of M is separable over K.

Problem 23.4. Let L/K be a separable field extension. Show that every element of L is separable over K.

We now address the uniqueness of the Galois closure.

Problem 23.5. Let K, L and M be as in the previous problem. Let $L \subseteq Q$ be a field extension such that Q/K is Galois. Show that there is an injection $M \hookrightarrow Q$ making the diagram

$$\begin{array}{ccccc} K & \subseteq & L & \subseteq & M \\ \parallel & & \parallel & & \downarrow \\ K & \subseteq & L & \subseteq & Q \end{array}$$

commute.

Problem 23.6. Let L/K be a separable field extension. Let $\theta_1, \ldots, \theta_N$ be a list of separable elements generating L over K, and let $\widehat{\theta}_1, \ldots, \widehat{\theta}_{\widehat{N}}$ be another such list. Let $g_j(x)$ be the minimal polynomial of θ_j over K and let $\widehat{g}_j(x)$ be the minimal polynomial of $\widehat{\theta}_j$. Let M be the splitting field of $\prod g_j(x)$ and let $\widehat{M}$ be the splitting field of $\prod \widehat{g}_j(x)$. Show that $M \cong \widehat{M}$.

So Galois closures are unique up to isomorphism.

Finally, we address separability in characteristic p.

Definition Let k be a field of characteristic p. We define k to be ***perfect*** if every element of k has a p-th root. We also define all fields of characteristic zero to be perfect.

The next problem was on the problem sets, check that your whole group knows how to do it:

Problem 23.7. Show that finite fields are perfect.

Problem 23.8. Let k be a perfect field of characteristic p and let $f(x) \in k[x]$. Show that, if the derivative $f'(x)$ is 0, then $f(x) = g(x)^p$ for $g(x) \in k[x]$.

Problem 23.9. Let k be a perfect field and let $f(x) \in k[x]$ be an irreducible polynomial. Show that $f(x)$ is separable.

Problem 23.10. Show that, if K is perfect then every finite degree extension of K is separable.

The simplest example of a nonperfect field is $\mathbb{F}_p(t)$.