The following problem was on the problem sets, check that everyone knows how to solve it:

**Problem 24.1.** Let $L$ be a field, let $H$ be a group of automorphisms of $L$ and let $F = \mathrm{Fix}(H)$, the elements of $L$ fixed by $H$. Suppose that $V$ is an $L$-vector subspace of $L^n$ and that $H$ takes $V$ to itself. Show that $V$ has a basis whose elements lie in $F^n$.

**One of several results called Artin's Lemma:** Let $L$ be a field, let $H$ be a finite group of automorphisms of $L$ and let $F = \mathrm{Fix}(H)$, the elements of $L$ fixed by $H$. Then $[L : F] = \#(H)$ and $H = \mathrm{Aut}(L/F)$.

Throughout this worksheet, let $L$, $H$ and $F$ be as above.

**Problem 24.2.** Show that $\#(H) \leq [L : F]$. This is just quoting something you've already done.

Suppose for the sake of contradiction that there are $n > \#(H)$ elements $\alpha_1, \alpha_2, \ldots, \alpha_n \in L$ which are linearly independent over $F$. Define

$$V = \left\{ (c_1, c_2, \ldots, c_n) \in L^n \ : \ \sum_j c_j h(\alpha_j) = 0 \ \forall h \in H \right\}.$$

**Problem 24.3.** Show that $V$ is an $L$-vector subspace of $L^n$ and that $H$ takes $V$ to itself.

**Problem 24.4.** Show that $\dim_L V > 0$.

**Problem 24.5.** Deduce a contradiction, and explain why you have proved $[L : F] = \#(H)$.

**Problem 24.6.** Show that $H = \mathrm{Aut}(L/F)$.

Artin's Lemma gives us a wide source of Galois extensions:

**Problem 24.7.** Let $L$, $H$ and $F$ be as in Artin's Lemma. Show that $[L : F]$ is Galois.