# 25. The Galois correspondence

Recall:

> **Theorem/Definition** Let $L/K$ be a field extension of finite degree. The following are equivalent:
> 
> (1) We have $\# \operatorname{Aut}(L/K) = [L : K]$.
> (2) The fixed field of $\operatorname{Aut}(L/K)$ is $K$.
> (3) For every $\theta \in L$, the minimal polynomial of $\theta$ over $K$ is separable and splits in $L$.
> (4) $L$ is the splitting field of a separable polynomial $f(x) \in K[x]$.
> 
> A field extension $L/K$ which satisfies these equivalent definitions is called ***Galois***.

Given a subfield $F$ with $K \subseteq F \subseteq L$, we write $\operatorname{Stab}(F)$ for the subgroup of $G$ fixing $F$; given a subgroup $H$ of $\operatorname{Gal}(L/K)$, we write $\operatorname{Fix}(H)$ for the subfield of $L$ fixed by $H$. Our next main goal will be to show:

> **The fundamental Theorem of Galois theory** Let $L/K$ be a Galois extension with Galois group $G$. The maps $\operatorname{Stab}$ and $\operatorname{Fix}$ are inverse bijections between the set of subgroups of $G$ and the set of intermediate fields $F$ with $K \subseteq F \subseteq L$. Moreover, if $F_1 \subseteq F_2$, then $\operatorname{Stab}(F_1) \supseteq \operatorname{Stab}(F_2)$ and $[\operatorname{Stab}(F_1) : \operatorname{Stab}(F_2)] = [F_2 : F_1]$. If $H_1 \subseteq H_2$ then $\operatorname{Fix}(H_1) \supseteq \operatorname{Fix}(H_2)$ and $[\operatorname{Fix}(H_1) : \operatorname{Fix}(H_2)] = [H_2 : H_1]$.

We start by proving some basic results about Fix and $\operatorname{Stab}$.

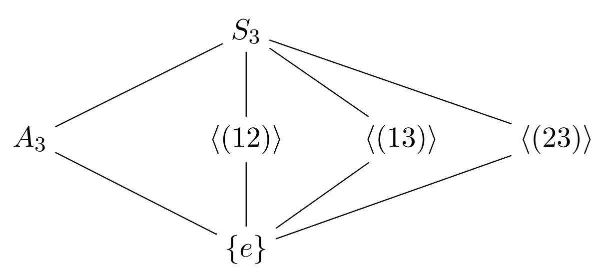
**Problem 25.1.** Let $L/K$ be Galois and let $F$ be a field with $K \subseteq F \subseteq L$. Show that $L/F$ is Galois and identify $\operatorname{Gal}(L/F)$ with a subgroup of $\operatorname{Gal}(L/K)$.

**Problem 25.2.**
(1) Show that, if $F_1 \subseteq F_2$ then $\operatorname{Stab}(F_1) \supseteq \operatorname{Stab}(F_2)$.
(2) Show that, if $H_1 \subseteq H_2$ then $\operatorname{Fix}(H_1) \supseteq \operatorname{Fix}(H_2)$.

**Problem 25.3.**
(1) Show that $\operatorname{Stab}(\operatorname{Fix}(H)) \supseteq H$.
(2) Show that $\operatorname{Fix}(\operatorname{Stab}(F)) \supseteq F$.

The Fundamental Theorem tells us that both of the $\supseteq$'s in Problem 25.3 are actually equality, but we don't know that yet.

We now give examples. Here is a table of the subgroups of $S_3$:




**Problem 25.4.** Let $L = \mathbb{Q}(x_1, x_2, x_3)$, let $S_3$ act on $L$ by permuting the variables and let $K = \operatorname{Fix}(S_3)$. Describe the subfield of $L$ fixed by each of the subgroups of $S_3$.

**Problem 25.5.** Let $L$ be the splitting field of $x^3 - 2$ over $\mathbb{Q}$. We number the roots of $x^3 - 2$ as $\sqrt[3]{2}$, $\omega \sqrt[3]{2}$ and $\omega^2 \sqrt[3]{2}$, where $\omega$ is a primitive cube root of $1$. Described the subfield of $L$ fixed by each of the subgroups of $S_3$.

Now we prove the theorem!

**Problem 25.6.** Let $L/K$ be a Galois extension. Let $F$ be a field with $K \subseteq F \subseteq L$.

(1) Show that $L/F$ is Galois.
(2) Show that $\operatorname{Aut}(L/F)$ is the subgroup $\operatorname{Stab}(F)$ of $\operatorname{Aut}(L/K)$.
(3) Show that $\operatorname{Fix}(\operatorname{Stab}(F)) = F$. Hint: What can you say about $[L : \operatorname{Fix}(\operatorname{Stab}(F))]$?

**Problem 25.7.** Let $L/K$ be a Galois extension with Galois group $G$. Let $H$ be a subgroup of $G$ and let $F = \operatorname{Fix}(H)$. Show that $\operatorname{Stab}(\operatorname{Fix}(H)) = H$.

**Problem 25.8.** Check the remaining claims of the Fundamental Theorem.