# 27. SOLVABLE EXTENSIONS

**Throughout this worksheet, let $F$ be a field of characteristic zero.**

**Problem 27.1.** Let $K$ be the splitting field of $x^n - 1$ over $F$. Show that $\mathrm{Gal}(K/F)$ is abelian.

**Problem 27.2.** Let $c \in F$ and let $K$ be the splitting field of $x^n - c$ over $F$. Show that $\mathrm{Gal}(K/F)$ is solvable.

A field extension $K/F$ is called **solvable** if there is a Galois extension $L/F$ with $K \subseteq L$ and $\mathrm{Gal}(L/F)$ solvable.

**Problem 27.3.** Let $K/F$ be a solvable extension. Let $K'$ be an extension of $K$ which is of the form $K[\theta]$ where $\theta^m \in K$ for some $\theta \in K'$. Show that $K'/F$ is solvable.

**Problem 27.4.** Let $F$ be a field and let $K_1/F$, $K_2/F$, $\ldots$, $K_r/F$ be solvable extensions of $F$. Show that there is a solvable extension $M$ of $F$ into which all the $K_j$ embed. (Hint: See Problem 26.3.)

**Problem 27.5.** (**The unsolvability of the quintic**) Let $f(x)$ be a degree $5$ separable polynomial in $F[x]$ and let $L$ be the splitting field of $f$ over $F$. Suppose that $\mathrm{Gal}(L/F)$ is $A_5$ or $S_5$. Show that $L$ is not contained in any solvable extension of $F$.

The point of the next problem is to drive home that we have completed the story of the quintic.

**Problem 27.6.** Let $f(x)$ be a degree $5$ separable polynomial in $\mathbb{Q}[x]$ and let $L$ be the splitting field of $f$ over $\mathbb{Q}$. Suppose that $\mathrm{Gal}(L/\mathbb{Q})$ is $A_5$ or $S_5$. Show that the roots of $f$ cannot be expressed in terms of rational numbers using $+$, $-$, $\times$, $\div$ and $\sqrt[m]{\ }$.