PROBLEM SET 2: DUE FRIDAY, JANUARY 22 BECAUSE OF MARTIN LUTHER KING DAY WEEKEND

Please see the course website for guidance on collaboration and formatting your problem sets.

Problem 2.1. Let p be a prime number and let G be a group of order p. Show that G is isomorphic to $\mathbb{Z}/p\mathbb{Z}$.

Problem 2.2. Let V be a vector space over some field. Let G be the set $V \times \bigwedge^2 V$ and define a multiplication operation on G by $(v, \alpha) * (w, \beta) = (v + w, \alpha + \beta + v \wedge w)$. Check that G is a group.

Problem 2.3. Let G be a group with n elements.

(1) Show that G is isomorphic to a subgroup of S_n.
(2) Let k be a field. Show that G is isomorphic to a subgroup of $\mathrm{GL}_n(k)$.

Problem 2.4. Let G be a group and let $g \in G$. The ***conjugacy class*** of g, written $\mathrm{Conj}(g)$, is $\{hgh^{-1} : h \in G\}$ and the ***centralizer*** of g, written $Z(g)$, is $\{h \in G : gh = hg\}$. Suppose that G is finite and let $g \in G$. Show that

$$\#(G) = \#\mathrm{Conj}(g) \cdot \#Z(g).$$

Problem 2.5. Let p be a prime and let G be a group of order p^k for some $k \geq 1$.

(1) Show that there is some $g \in G$ other than e such that $\mathrm{Conj}(g) = \{g\}$.
(2) Show that this g commutes with every $h \in G$.

Problem 2.6. Let G be a group and let H be a subgroup of G with $[G : H] = n$.

(1) Show that there is a normal subgroup N of G with $N \subseteq H \subseteq G$ and $[G : N]$ dividing $n!$.
(2) Suppose that n is prime and that $|G|$ is not divisible by any prime $< n$. Show that H is normal in G.

Problem 2.7. In this problem, we will check that A_n is simple for $n > 5$. You may assume that we already know A_5 is simple. Let N be a normal subgroup of A_n other than $\{e\}$. We will show that $N = A_n$.

Let $\sigma \in N$ with $\sigma \neq e$. Choose i with $\sigma(i) \neq i$ and choose $j \notin \{\sigma^{-1}(i), i, \sigma(i)\}$. Put $\tau = (i\sigma(i)j)$. Put $\gamma = \sigma^{-1}\tau^{-1}\sigma\tau$. Let X be a five[1] element subset of $\{1, 2, \ldots, n\}$ containing $\{\sigma^{-1}(i), i, \sigma(i), j, \sigma^{-1}(j)\}$. Let A_X be those permutations in A_n which fix all $x \notin X$.

(1) Show that $\gamma \in N$, that $\gamma \neq e$ and that $\gamma \in A_X$.
(2) Show that $N \supseteq A_X$. Hint: Notice that $N \cap A_X$ is normal in A_X.
(3) Show that N contains a permutation of the form (xyz).
(4) Show that $N = A_n$.

Problem 2.8. Let R be a ring. An R-module M is called ***simple*** if M has exactly two submodules, $\{0\}$ and M. Note that the zero module is *not* considered simple. The three parts of this problem are logically independent and can be done in any order, but share many techniques.

(1) (***Schur's lemma***) Let S be a simple R-module and let $\phi : S \to S$ be a nonzero R-linear homomorphism. Show that ϕ is invertible.
(2) Let S_1 and S_2 be two simple R-modules and let M be an R-submodule of $S_1 \oplus S_2$. Show that one of the following must hold: (a) $M = 0$ (b) $M = S_1 \oplus 0$ (c) $M = 0 \oplus S_2$ (d) $M = S_1 \oplus S_2$. or (e) $S_1 \cong S_2$ and $M = \{(x, \phi(x)) : x \in S_1\}$ for some isomorphism $\phi : S_1 \to S_2$.
(3) Let M be an R-module and let N_1 and N_2 be R-submodules such that M/N_1 and M/N_2 are simple. Show that either $N_1 = N_2$ or $M/(N_1 \cap N_2) \cong M/N_1 \oplus M/N_2$.

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[1] If $\sigma^{-1}(i)$, i, $\sigma(i)$, j and $\sigma^{-1}(j)$ are not distinct, add some extra elements to make $\#(X) = 5$.