2. A WEAK VERSION OF UNSOLVABILITY OF THE QUINTIC

> *Perhaps it will not be so difficult to prove, with all rigor, the impossibility for the fifth degree.*
>
> Karl Freidrich Gauss, 1799

One of the highlights of this course will be the proof of the unsolvability of the quintic. This worksheet proves a weaker version of this result.

Let L be the field of rational functions $\mathbb{C}(r_1, r_2, \dots, r_5)$. Define e_1, e_2, e_3, e_4, $e_5 \in L$ as the coefficients in

$$(x - r_1)(x - r_2)(x - r_3)(x - r_4)(x - r_5) = x^5 - e_1x^4 + e_2x^3 - e_3x^2 + e_4x - e_5.$$

> **Theorem** (Ruffini)**.** Starting from e_1, e_2, …, e_5, it is impossible to obtain the elements r_1, r_2, …, r_5 of L by the operations $+$, $-$, $\times$, $\div$, $\sqrt[n]{\ }$, **under the condition that**, every time we take an n-th root, we must stay in L.

Let S_5 act on L by permuting the r_i. Let K be the subfield of L fixed by S_5.

Problem 2.1. Show that the e_j are in K.

Problem 2.2. Set $\Delta = \prod_{i<j}(r_i - r_j)$. Show that $\Delta^2 \in K$ but $\Delta \notin K$.

We define A_5 to be the subgroup of S_5 fixing Δ. We will often write permutations using cycle notation: $(i_1 i_2 \cdots i_k)$ means the permutation which cycles $i_1 \mapsto i_2 \mapsto \cdots \mapsto i_k \mapsto i_1$ and fixes everything not in $\{i_1, i_2, \dots, i_k\}$.

Problem 2.3. Check that (123), (124) and (125) $\in A_5$.

Problem 2.4. Verify the following identities in the group A_5:

$$(123)^3 = (124)^3 = (125)^3 = \mathrm{Id} \qquad ((123)(124))^2 = ((123)(125))^2 = ((124)(125))^2 = \mathrm{Id}\,.$$

Problem 2.5. Show that there are no nontrivial group homomorphisms from A_5 to an abelian group. You may assume that (123), (124) and (125) generate A_5; you'll check this on the problem set.

Let F be the subfield of L fixed by A_5.

Problem 2.6. Suppose that $f \in L$ is nonzero, and $f^n \in F$. For $\sigma \in A_5$, show that $\frac{\sigma(f)}{f} \in \mathbb{C}^\times$.

Problem 2.7. Let f be as in Problem 2.6. For $\sigma \in A_5$, define $\chi_f(\sigma) = \frac{\sigma(f)}{f}$. Show that $\chi_f : A_5 \to \mathbb{C}^\times$ is a group homomorphism.

Problem 2.8. Show that, if $f \in L$ and $f^n \in F$, then $f \in F$.

Problem 2.9. Prove Ruffini's Theorem.

History, and plan of the course: Paolo Ruffini, *Teoria generale delle equazioni*, 1799 gave what, in modern language, is a proof of this result. His work was difficult to understand, and the assumption that one would not leave $\mathbb{C}(r_1, \dots, r_5)$ when extracting n-th roots was only stated implicitly. Calling this result Ruffini's Theorem is not standard, but seems appropriate.

Abel replaced the use of rational functions with multivalued complex analytic functions of the r_j. He proved the corresponding result with no restriction on taking roots in 1824, just four years prior to his death of tuberculosis at age 26. Because the foundations of complex analysis were not yet settled, his work was also hard to follow. He died only four years later of tuberculosis. The unsolvability of the quintic is now known as the Abel-Ruffini Theorem.

Neither Abel nor Ruffini was able to prove that the roots of a particular quintic with, for example, rational coefficients, were not expressible in terms of the coefficients of that quintic; that would wait for Galois in 1831, just a year before his death in a duel at age 20. Our course will follow the ideas of Galois.

We can see from these early attempts that the lack of a notion of adjoining an n-th root to an arbitrary field, without some ambient field to work inside, was a major obstacle to clear proofs. For this reason, we will be studying abstract fields. That will require replacing the groups S_n and A_n with general abstract groups, so first we will study groups.

> *Abel has left mathematicians enough to keep them busy for 500 years.* Atttributed to Charles Hermite.