For most of this class, we have discussed field extensions of finite degree. We will now discuss a notion of "size" for extensions of infinite degree.

Let K/k be a field extension and let $\theta_1, \ldots, \theta_r \in K$.

Definition: We say that θ_1, ..., θ_r are ***algebraically independent over*** k if, for all nonzero polynomials $f(t_1, \ldots, t_r) \in k[t_1, \ldots, t_r]$, we have $f(\theta_1, \ldots, \theta_r)$ nonzero.

Definition: The ***algebraic span of*** θ_1, $\ldots \theta_r$ ***in*** K ***over*** k is the set of those $\phi \in K$ which are algebraic over $k(\theta_1, \ldots, \theta_r)$. We say that $\theta_1, \ldots, \theta_r$ is an ***algebraic spanning set for*** K ***over*** k if every $\phi \in K$ is algebraic over $k(\theta_1, \ldots, \theta_r)$.

Definition: We say that $\theta_1, \ldots, \theta_r$ is a ***transcendence basis for*** K ***over*** k if $\theta_1, \ldots \theta_r$ is both algebraically independent and an algebraic spanning set.

The analogy with linear algebra should be clear. We start by showing the analogue of "finitely generated vector spaces have bases".

Problem 30.1. Suppose that $\theta_1, \ldots, \theta_r$ is an algebraic spanning set for K over k. Show that there is some subset of $\{\theta_1, \ldots, \theta_r\}$ which is a transcendence basis for K over k.

We now prove some useful lemmas:

Problem 30.2. Let $\theta_1, \ldots, \theta_r \in K$ and let $0 < q < r$. Show that $\theta_1, \ldots, \theta_r$ are an algebraic spanning set over k if and only if $\theta_{q+1}, \ldots, \theta_r$ are an algebraic spanning set over $k(\theta_1, \ldots, \theta_q)$.

Problem 30.3. Let $\theta_1, \ldots, \theta_r \in K$ and let $0 < q < r$. Show that $\theta_1, \ldots, \theta_r$ are algebraically independent over k if and only if the following two conditions hold:

- $\theta_1, \ldots, \theta_q$ are algebraically independent over k and
- $\theta_{q+1}, \ldots, \theta_{q+1}$ are algebraically independent over $k(\theta_1, \ldots, \theta_q)$.

We can now prove the analogue of "linearly independent sets can be extended to bases":

Problem 30.4. Let $\theta_1, \theta_2, \ldots, \theta_r$ be algebraically independent over k. Let $\phi_1, \ldots, \phi_s$ be an algebraic spanning set for K over k. Show that there is some subset S of $\{\phi_1, \ldots, \phi_s\}$ such that $\{\theta_1, \ldots, \theta_r\} \cup S$ is a transcendence basis for K over k.

We now start in on proving that all transcendence bases have the same size.

Problem 30.5. Let $\alpha_1, \alpha_2, \ldots, \alpha_p$ be an algebraic spanning set for K over k. Let $\beta \in K$ be not algebraic over k. Show that there is some index j such that $\alpha_1, \alpha_2, \ldots, \alpha_{j-1}, \beta, \alpha_{j+1}, \ldots, \alpha_p$ is an algebraic spanning set for K over k.

Problem 30.6. Let $\alpha_1, \alpha_2, \ldots, \alpha_p$ be an algebraic spanning set for K over k and let $\beta_1, \ldots, \beta_q$ be algebraically independent over k. Show that $p \geq q$. Hint: Induct on the number of elements of $\{\beta_1, \ldots, \beta_q\}$ which are not in $\{\alpha_1, \ldots, \alpha_p\}$.

Problem 30.7. Show that any two finite transcendence bases of K over k have the same size. This size is called the ***transcendence degree*** of K over k.

Remark: This worksheet is deliberately written to avoid the Axiom of Choice. If you are comfortable with the Axiom of Choice, then we can define infinite transcendence bases in the obvious way and show that every field extension has a transcendence basis, and that any two transcendence bases have the same cardinality.