PROBLEM SET 3: DUE WEDNESDAY, JANUARY 29

Problem 3.1. Describe all actions of C_2 (by group homomorphisms) on the group C_{21}.

Problem 3.2. ~~Let G be a group and let g be an element of G. Define an action of $\mathbb{Z}$ on G by $\phi(k)(h) = g^k h g^{-k}$. Show that $G \rtimes_\phi \mathbb{Z} \cong G \times \mathbb{Z}$. This gives an example of how two different actions can give isomorphic semi-direct products.~~ (**Problem cancelled because we won't get to semidirect products soon enough.**)

Problem 3.3. Does there exist a group G with normal subgroups N_1 and N_2 such that $N_1 \cong S_5$, $N_2 \cong S_7$, $G/N_1 \cong S_{42}$ and $G/N_2 \cong S_{41}$?

Problem 3.4. Let G be a group and N a normal subgroup. Let $g \in N$ and recall that $Z_G(g) = \{h \in G : gh = hg\}$.

(1) Let $h \in G$, so g and hgh^{-1} are both in N and are conjugate in G. Show that hgh^{-1} is conjugate to g in N if and only if $h \in Z_G(g)N$.
(2) Show that the S_5-conjugacy class of (123) is also an A_5-conjugacy class, but that the S_5-conjugacy class of (12345) splits into two A_5-conjugacy classes.
(3) For each of the following elements of $\mathrm{SL}_2(\mathbb{R})$, describe how the $\mathrm{GL}_2(\mathbb{R})$ conjugacy class splits into conjugacy classes of $\mathrm{SL}_2(\mathbb{R})$:
$$\begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Problem 3.5. Let G be a group and H a subgroup. The subgroup H is called ***canonical*** if $\phi(H) = H$ for every automorphism ϕ of G.

(1) Show that, if H is canonical in G, then H is normal in G.

Let A be a subgroup of B, which is a subgroup of C. For each of these statements, give a proof or counterexample:

(2) If A is canonical in B and B is canonical in C, then A is canonical in C.
(3) If A is canonical in B and B is normal in C, then A is normal in C.
(4) If A is normal in B and B is canonical in C, then A is normal in C.
(5) If A is normal in B and B is normal in C, then A is normal in C.

Problem 3.6. Let $1 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 1$ be a short exact sequence of groups.

(1) A ***left splitting***[1] of this sequence is a map $\lambda : B \to A$ with $\lambda \circ \alpha = \mathrm{Id}_A$. Show that, if the sequence $1 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 1$ has a left splitting, then $B \cong A \times C$.
(2) ~~A ***right splitting*** of this sequence is a map $\rho : C \to B$ such that $\beta \circ \rho = \mathrm{Id}_C$. Show that, if the sequence $1 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 1$ has a right splitting, then $B \cong A \rtimes C$ for an action of A on C. Hint: Apply Worksheet Problem 8.6 to the subgroups $\alpha(A)$ and $\rho(C)$ of B.~~ (**Problem cancelled because we won't get to semidirect products soon enough.**)

Problem 3.7. Let R be a ring and let M be an R-module. A ***Jordan-Holder filtration*** of M is a chain of submodules $0 = M_0 \subset M_1 \subset \cdots \subset M_\ell = M$ such that M_k/M_{k-1} is simple[2] for $1 \leq k \leq \ell$. A module which has a Jordan-Holder filtration is said to ***have finite length***.

(1) This problem was on the 593 problem sets, but please write it out to check that you remember it: Let M have finite length and let N be a submodule of M. Show that N and M/N have finite length.

In this problem, we will prove the ***Jordan-Holder Theorem for modules*** which states: Let $0 = M_0 \subset M_1 \subset \cdots \subset M_\ell = M$ and $0 = M'_0 \subset M'_1 \subset \cdots \subset M'_{\ell'} = M$ be two Jordan-Holder filtrations. Then $\ell = \ell'$ and there is a permutation σ of $\{1, 2, \ldots, \ell\}$ such that $M_k/M_{k-1} \cong M'_{\sigma(k)}/M'_{\sigma(k)-1}$. Our proof is by induction on $\min(\ell, \ell')$.

(2) Do the base case $\min(\ell, \ell') = 1$.
(3) Explain why we are done if $M_{\ell-1} = M'_{\ell'-1}$.
(4) Suppose that $M_{\ell-1} \neq M'_{\ell'-1}$ and put $N = M_{\ell-1} \cap M'_{\ell'-1}$. Explain why we are also done in this case.

[1]When I google "left splitting", Google's first suggestion is "left splitting headache". I hope this will not be your opinion!

[2]Look back to Problem 2.8 for previous results on simple modules. You may use any of the parts of that problem in this one.