Definition. A ***group*** G is a set with a binary operation $* : G \times G \to G$ obeying the properties

(1) There is an element 1 of G such that $1 * g = g * 1 = g$ for all $g \in G$.
(2) For all $g \in G$, there is an element g^{-1} obeying $g * g^{-1} = g^{-1} * g = 1$.
(3) For all $g_1, g_2, g_3 \in G$, we have $(g_1 * g_2) * g_3 = g_1 * (g_2 * g_3)$.

Given a group G, a ***subgroup*** of G is a subset containing 1 and closed under $*$ and $g \mapsto g^{-1}$.

Depending on context, we may denote $*$ by $*$, $\times$, $\cdot$ or no symbol at all, and we may denote 1 as 1, e or Id.

Problem 3.1. Show that a group G only has one element 1 obeying the condition (1).

Problem 3.2. Let G be a group and let $g \in G$. Show that G only has one element obeying the condition (2).

Definition. Given two groups G and H, a ***group homomorphism*** is a map $\phi : G \to H$ obeying $\phi(g_1 * g_2) = \phi(g_1) * \phi(g_2)$. A bijective group homomorphism is called an ***isomorphism*** and two groups are called ***isomorphic*** if there is an isomorphism between them.

A group homomorphism can also be called a "map of groups" or a "group map".

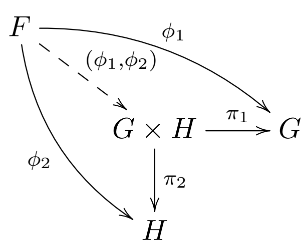
Problem 3.3. Let $\phi : G \to H$ be a group homomorphism. Show that $\phi(1) = 1$ and $\phi(g^{-1}) = \phi(g)^{-1}$.

Problem 3.4. Let $\phi : G \to H$ be a group homomorphism.

(1) The ***image*** of ϕ is $\mathrm{Im}(\phi) := \{\phi(g) : g \in G\}$. Show that $\mathrm{Im}(\phi)$ is a subgroup of G.
(2) The ***kernel*** of ϕ is $\mathrm{Ker}(\phi) := \{g \in G : \phi(g) = 1\}$. Show that $\mathrm{Ker}(\phi)$ is a subgroup of G.

Definition. Given two groups G and H, the ***product group*** is the group whose underlying set is $G \times H$, with multiplication structure $(g_1, h_1) * (g_2, h_2) = (g_1 g_2, h_1 h_2)$.

Problem 3.5. Let G and H be two groups and let π_1 and π_2 be the projections $G \times H \to G$ and $G \times H \to H$ onto the first and second factor. Show that $G \times H$ obeys the ***universal property of products***, meaning that, for any group F with maps $\phi_1 : F \to G$ and $\phi_2 : F \to H$, there is a unique map $(\phi_1, \phi_2) : F \to G \times H$ such that the diagram below commutes:



Definition. A group G is called ***abelian*** if $g_1 * g_2 = g_2 * g_1$ for all $g_1, g_2 \in G$.

If G is abelian, we will often denote $*$ by $+$ and 1 by 0. We will **never** use these notations for a non-abelian group.

Problem 3.6. Let G be a group. Show that G is abelian if and only if:

(1) The map $g \mapsto g^{-1}$ is a group homomorphism.
(2) The map $g \mapsto g^2$ is a group homomorphism.
(3) The map $\mu : G \times G \to G$ by $\mu(g, h) = g * h$ is a group homomorphism.

We'll toss in one more definition:

Definition. For $g \in G$, the ***conjugacy class*** of g is the set $\mathrm{Conj}(g) := \{hgh^{-1} : h \in G\}$.