PROBLEM SET 4: DUE WEDNESDAY, FEBRUARY 5

Please see the course website for guidance on collaboration and formatting your problem sets.

Problem 4.1. Let $1 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 1$ be a short exact sequence of groups. A ***right splitting*** of this sequence is a map $\rho : C \to B$ such that $\beta \circ \rho = \mathrm{Id}_C$. Show that, if the sequence $1 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 1$ has a right splitting, then $B \cong A \rtimes C$ for an action of A on C. Hint: Apply Worksheet Problem 8.6 to the subgroups $\alpha(A)$ and $\rho(C)$ of B.

Problem 4.2. Which of the following sequences are left split? Which are right split? (See Problems 3.6 and 4.1.)

(1) $1 \to C_2 \to C_6 \to C_3 \to 1$.
(2) $1 \to C_2 \to C_4 \to C_2 \to 1$.
(3) $1 \to A_5 \to S_5 \to \{\pm 1\} \to 1$.
(4) $1 \to \mathrm{SL}_3(\mathbb{R}) \to \mathrm{GL}_3(\mathbb{R}) \xrightarrow{\det} \mathbb{R}^\times \to 1$.

Problem 4.3. Let F be a field. Let $\mathrm{GL}_n(F)$ be the group of invertible $n \times n$ matrices with entries in F and let $\mathrm{SL}_n(F)$ be the group of submatrices with determinant 1. The aim of this problem is to describe the abelianization of $\mathrm{GL}_n(F)$ and $\mathrm{SL}_n(F)$ in all cases.

For $1 \leq i \neq j \leq n$ and $r \in F$, we define $E_{ij}(r)$ to be the matrix with ones on the diagonal, an r in position (i, j) and zeroes everywhere else; we call such a matrix an ***elementary matrix***. We showed in Math 593 (and you may use) that the elementary matrices generate $\mathrm{SL}_n(F)$.

(1) Suppose that $n \geq 3$. Show that the elementary matrices are in the commutator subgroup of $\mathrm{SL}_n(F)$. Conclude that the abelianization of $\mathrm{SL}_n(F)$ is trivial and the abelianization of $\mathrm{GL}_n(F)$ is $F^\times$. Hint: First think about matrices of the form $\left[\begin{smallmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{smallmatrix}\right]$.
(2) Show that the elementary matrices are in the commutator subgroup of $\mathrm{GL}_2(F)$ for $\#(F) > 2$ and in the commutator subgroup of $\mathrm{SL}_2(F)$ for $\#(F) > 3$. Conclude that the abelianization of $\mathrm{SL}_2(F)$ is trivial and the abelianization of $\mathrm{GL}_2(F)$ is $F^\times$ in these cases. Hint: First think about matrices of the form $\left[\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right]$.
(3) What are the abelianizations of $\mathrm{GL}_2(\mathbb{F}_2) = \mathrm{SL}_2(\mathbb{F}_2)$ and $\mathrm{SL}_2(\mathbb{F}_3)$?

Problem 4.4. Let A be an abelian group and let $\phi : A \to A$ be an automorphism. Let $\mathbb{Z}$ act on A by $k : a \mapsto \phi^k(a)$; by a standard abuse of notation, we will also denote this action by ϕ. Let $G = A \rtimes_\phi \mathbb{Z}$. Show that the abelianization of G is isomorphic to $A/(\mathrm{Id} - \phi)(A) \times \mathbb{Z}$. In this formula, $\mathrm{Id} - \phi$ is an additive map $A \to A$ and we are quotienting by its image.

The remaining problems are not tightly tied to the current material; but many of them will be useful in the future.

Problem 4.5. Let G be a finite group.

(1) Let X be a finite set with a transitive action of G, and $|X| > 1$. Show that there is some $g \in G$ which fixes no element of X. Hint: What lemma have we proved involving the number of fixed points?
(2) Let $H \subsetneq G$ be a proper subgroup of G. Show that there is some conjugacy class C of G with $H \cap C = \emptyset$.

Problem 4.6. Let p be a prime number. For a positive integer n, let $v(n)$ be the integer such that p divides $n!$ precisely $v(n)$ times.

(1) Write $n = mp + r$ for $0 \leq r \leq p - 1$. Show that $v(n) = m + v(m)$.
(2) Show that S_n contains a subgroup of order $p^{v(n)}$. Hint: You might find it a good warm up to do the cases $n = mp$ for $m < p$ and $n = p^2$ first.

Problem 4.7. Let k be a finite field with q elements and let $1 \leq m \leq n$. Show that the number of $m \times n$ matrices with entries in k and rank m is $\prod_{j=0}^{m-1}(q^n - q^j)$. (Hint: Induct on m.)

Problem 4.8. Let R be a ring (not assumed commutative). An element $x \in R$ is called ***nilpotent*** if there is some positive integer m for which $x^m = 0$. A (two-sided) ideal whose every element is nilpotent is called ***nil***.

(1) Show that, if x is nilpotent, then $1 + x$ is a unit.
(2) Let N be a nil ideal of R. Let $U = \{1 + x : x \in N\}$. Show that U is a group under multiplication.