## 4. Group actions

**Definition.** Let $G$ be a group and let $X$ be a set. An ***action*** of $G$ on $X$ is a map $* : G \times X \to X$ obeying $(g_1 * g_2) * x = g_1 * (g_2 * x)$ and $e * x = x$.

Depending on context, we may denote $*$ by $*$, $\times$, $\cdot$ or no symbol at all. This notion of an action can also be called a "left action"; a "right action" is a map $* : X \times G \to X$ obeying $x * (g_2 * g_1) = (x * g_2) * g_1$.

**Problem 4.1.** Let $G \times X \to X$ be a left action of $G$ on $X$. Define a map $X \times G \to X$ by $(x, g) \mapsto g^{-1}x$. Show that this is a right action of $G$ on $X$.

**Problem 4.2.** Let $S_X$ be the group of bijections $X \to X$, with the group operation of composition. Show that an action of $G$ on $X$ is the same as a group homomorphism $G \to S_X$.

**Definition.** Let $G$ be a group which acts on a set $X$. For $x \in X$, the ***stabilizer*** $\mathrm{Stab}(x)$ of $x$ is $\{g \in G : g * x = x\}$. For $g \in G$, the ***fixed points*** $\mathrm{Fix}(g)$ of $g$ are $\{x \in X : g * x = x\}$.

**Problem 4.3.** With $G$, $X$ and $x$ as above, show that $\mathrm{Stab}(x)$ is a subgroup of $X$.

**Problem 4.4.** Let $G$, $X$ and $x$ be as above and let $g \in G$. Show that $\mathrm{Stab}(gx) = g\,\mathrm{Stab}(x)g^{-1}$.

**Definition.** For $G$, $X$ and $x$ as above, the ***orbit*** of $x$, written $Gx$, is $\{gx : g \in G\}$.

**Problem 4.5.** (**The Orbit-Stabilizer theorem**) If $G$ is finite, show that $\#(G) = \#(Gx)\#(\mathrm{Stab}(x))$.

The set of orbits of $G$ on $X$ is denoted $G\backslash X$. If we have a right action, we write $X/G$.

**Problem 4.6.** (**Burnside's Lemma**[1]) Let $G$ be a finite group and let $X$ be a finite set on $G$ acts. Show that

$$\frac{1}{\#G}\sum_{g \in G} \#\mathrm{Fix}(g) = \#(G\backslash X).$$

**Definition.** Let $G$ be a group and let $H$ be a subgroup. Let $H$ act on $G$ by $h * g = hg$. The orbits of this action are called the ***right cosets*** of $H$ in $G$. The ***left cosets*** are the orbits for the right action $G * H \to G$. The number of cosets of $H$ in $G$ is called the ***index*** of $H$ in $G$ and written $[G : H]$.

**Problem 4.7.** Show that $G$ has a left action on the set $G/H$ of left cosets, such that $g_1 * (g_2 H) = (g_1 * g_2)H$. Show that the stabilizer of the coset $eH$ is $H$.

**Problem 4.8.** (**Lagrange's Theorem**[2]) Let $G$ be a finite group and let $H$ be a subgroup. Show that $\#(H)$ divides $\#(G)$.

**Problem 4.9.** Let $G$ be a finite group with $\#(G) = N$. Let $g \in G$ and let the group generated by $g$ have $n$ elements.

(1) Show that $n$ divides $N$.
(2) Show that $g^N = 1$.

[1] Proved by Ferdinand Georg Frobenius.

[2] Proved by Camille Jordan.