Please see the course website for guidance on collaboration and formatting your problem sets.

Problem 5.1. Recall that $Z(G)$ is the center of a group G, and that a ***central series*** of G is a sequence of subgroups $G_0 \lhd G_1 \lhd G_2 \lhd \cdots \lhd G_N$ where $G_{i+1}/G_i \subseteq Z(G/G_i)$ for all i.

(1) The ***upper (or ascending) central series*** of G is defined inductively by $U_0 = \{e\}$ and $U_{k+1} = \pi_k^{-1}(Z(G/U_k))$, where π_k is the projection $G \to G/U_k$. Let $\{e\} = G_0 \lhd G_1 \lhd G_2 \lhd \cdots$ be a central series with $G_0 = \{e\}$. Show that $G_k \subseteq U_k$.
(2) The ***lower (or descending) central series*** is defined inductively by $L^0 = G$ and letting L^{k+1} be the group generated by all products $ghg^{-1}h^{-1}$ with $g \in G$ and $h \in L^k$. Let $G = G^0 \rhd G^1 \rhd G^2 \rhd \cdots$ be a central series with $G^0 = G$ (note that we have reversed the direction of the numbering). Show that $L^k \subseteq G^k$.
(3) Recall that a group G is called ***nilpotent*** if there is a central series $\{e\} = G_0 \lhd G_1 \lhd G_2 \lhd \cdots \lhd G_N = G$. Show that G is nilpotent, if and only if U_k is eventually G, if and only if L^k is eventually $\{e\}$.

Problem 5.2. Let R be a ring (not assumed commutative) and let I be a two sided ideal of R. We define I^m to be the two sided ideal generated by all products $x_1 x_2 \cdots x_m$ for $x_1, x_2, \ldots, x_m \in I$. We define the ideal N to be ***nilpotent*** if there is a positive integer m such that $N^m = (0)$. Let N be a nilpotent ideal and let U be the group $\{1 + x : x \in N\}$. Show that U is a nilpotent group.[1]

The next three problems are adapted from QR exams.

Problem 5.3. Let G be a group where $g^2h^2 = h^2g^2$ for all g and h.

(1) Let N be the subgroup $\langle g^2 \mid g \in G \rangle$. Show that N is normal.
(2) Show that G is solvable.

Problem 5.4. A group G is called ***virtually solvable*** if G has a normal subgroup N such that N is solvable and G/N is finite.

(1) Show that a subgroup of a virtually solvable group is virtually solvable.
(2) Show that a quotient of a virtually solvable group is virtually solvable.

Problem 5.5. Let G_1 and G_2 be groups and let S be a subgroup of $G_1 \times G_2$. Let H_i be the projection of S onto G_i and let $K_i = S \cap G_i$.

(1) Show that K_i is normal in H_i.
(2) Show that $H_1/K_1 \cong H_2/K_2$.

These problems do not use the current material, but will be useful in the future. Let k be a field and let $k[x]$ be the ring of polynomials with coefficients in x. Recall that $k[x]$ is a PID; you may use this fact freely in these problems.

Problem 5.6. Let $K \subset L$ be two fields and let $a(x)$ and $b(x) \in K[x]$. Let $g(x)$ be the GCD of a and b in $K[x]$. Show that g is also the GCD of a and b in $L[x]$.

Problem 5.7. For a polynomial $f(x) = \sum f_j x^j \in k[x]$, we define the derivative $f'(x)$ to be $\sum j f_j x^{j-1}$.

(1) For any two polynomials $f(x)$ and $g(x) \in k[x]$, show that $(f+g)'(x) = f'(x) + g'(x)$ and $(fg)'(x) = f(x)g'(x) + f'(x)g(x)$. Note that your proof should work for any field k.

For a nonzero polynomial $f(x)$ and an irreducible polynomial $p(x)$, let $m_p(f)$ be the number of times that p appears in the prime factorization of f.

(2) Let f and p be as above and suppose that $m_p(f) > 0$. Show that, if k has characteristic zero, then $m_p(f') = m_p(f) - 1$.
(3) Give an example to show that the above need not be true in nonzero characteristic.

[1]This is the most direct conceptual connection I know between the uses of the word "nilpotent" in ring theory and in group theory. The actual historical origins of the word come from the following related case: Start with R being $n \times n$ upper-triangular real matrices, N being upper-triangular real matrices with 0's on the diagonal and U being upper-triangular real matrices with 1's on the diagonal. Let G be a Lie subgroup of U, then the Lie algebra $\mathfrak{g}$ is a Lie subalgebra of N. The group G is a nilpotent group, and the Lie algebra $\mathfrak{g}$ is a nilpotent Lie algebra. (We have not defined Lie groups, Lie algebras or what it means for a Lie algebra to be nilpotent.)