5. Normal subgroups, quotient groups, short exact sequences

Problem 5.1. Let G be a group and let N be a subgroup. Show that the following are equivalent:

(1) For all $g \in G$, we have $gNg^{-1} = N$.
(2) All elements of G/N have the same stabilizer, for the left action of G on G/N.
(3) Every left coset of N in G is also a right coset.
(4) If $g_1 N = g_1' N$ and $g_2 N = g_2' N$, then $g_1 g_2 N = g_1' g_2' N$.

Definition. A subgroup N obeying the equivalent conditions of Problem 5.1 is called a ***normal subgroup*** of G. We write $N \trianglelefteq G$ to indicate that N is a normal subgroup of G.

Problem 5.2. Let G be S_3. Which of the following subgroups are normal?

(1) The subgroup generated by (12).
(2) The subgroup generated by (123).

Problem 5.3. Let G be a group and let N be a normal subgroup of G.

(1) Prove or disprove: Let $\alpha : F \to G$ be a group homomorphism. Then $\alpha^{-1}(N)$ is normal in F.
(2) Prove of disprove: Let $\beta : G \to H$ be a group homomorphism. Then $\beta(N)$ is normal in H.
(3) At least one of the statements above is false. Find an additional hypothesis you could add to make it true.

Definition. Given a group G and an normal subgroup N, the ***quotient group*** G/N is the group whose underlying set is the set of cosets G/N with multiplication such that $(g_1 N)(g_2 N) = g_1 g_2 N$.

This definition makes sense by Part (4) of Problem 5.1. I won't make you check that this is a group, but do so on your own time if you have any doubt. Also, I won't make you check this, but the groups G/N and $N \backslash G$, defined in the obvious ways, are isomorphic.

Let $\phi : G \to H$ be a group homomorphism. Recall that the image and kernel of ϕ are $\mathrm{Ker}(\phi) := \{g \in G : \phi(g) = 1\}$ and $\mathrm{Im}(\phi) := \{\phi(g) : g \in G\}$.

Problem 5.4. Show that the kernel of ϕ is a normal subgroup of G.

Problem 5.5. Show that the "obvious" map from $G/\mathrm{Ker}(\phi)$ to $\mathrm{Im}(\phi)$ is an isomorphism.

We often discuss quotients using the language of short exact sequences:

Definition. A ***short exact sequence*** $1 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 1$ is three groups A, B and C, and two group homomorphisms $\alpha : A \to B$ and $\beta : B \to C$ such that α is injective, β is surjective, and $\mathrm{Im}(\alpha) = \mathrm{Ker}(\beta)$.

I will occasionally write 0 instead of 1 at one end or the other of a short exact sequence. I do this when the adjacent group (meaning A or C) is abelian and it would feel bizarre to denote the identity of that abelian group as 1.

We'll write C_n for the abelian group $\mathbb{Z}/n\mathbb{Z}$. This is called the ***cyclic group*** of order n.

Problem 5.6. Show that there is a short exact sequence $1 \to C_m \to C_{mn} \to C_n \to 1$.

Problem 5.7. Show that there is a short exact sequence $1 \to C_3 \to S_3 \to S_2 \to 1$.

Problem 5.8. Show that there is a short exact sequence $1 \to C_2^2 \to S_4 \to S_3 \to 1$.

Problem 5.9. What is the relationship between Problems 5.7 and 5.8 and your computations on the first day of class involving $\{(\beta_1 + \omega\beta_2 + \omega^2\beta_3)^3,\ (\beta_1 + \omega^2\beta_2 + \omega\beta_3)^3\}$ and $\{(\gamma_1 + \gamma_2 - \gamma_3 - \gamma_4)^2,\ (\gamma_1 - \gamma_2 + \gamma_3 - \gamma_4)^2,\ (\gamma_1 - \gamma_2 - \gamma_3 + \gamma_4)^2\}$?