PROBLEM SET 6: DUE WEDNESDAY, FEBRUARY 19

Please see the course website for guidance on collaboration and formatting your problem sets.

Problem 6.1. A standard rite of passage is to check that there are no non-abelian simple groups of order < 60, so let's do that. Let G be a non-abelian simple group.

(1) Show that the order of G is not a prime power.
(2) Show that, for every prime p dividing $\#(G)$, there must be some n_p dividing $\#(G)$ with $n_p > 1$ and $n_p \equiv 1 \bmod p$.

At this point, we have ruled out all cases except 12, 24, 30, 36, 48 and 56.

(3) In the notation of the previous problem, show that furthermore we must have $\#(G) | \frac{n_p!}{2}$.

This rules out 12, 24, 48 (take $p = 2$) and 36 (take $p = 3$).

(4) Suppose that G were a simple group of order 30. Show that G would contain 24 elements of order 5 and 20 elements of order 3; deduce a contradiction.
(5) Suppose that G were a simple group of order 56. Show that G would contain 48 elements of order 7 and > 8 elements whose order is a power of 2; deduce a contradiction.

Problem 6.2. In the previous problem, you checked that there are no non-abelian simple groups of order < 60. Explain why this means that every group of order < 60 is solvable.

Problem 6.3. Let n be a positive integer. Show that a group of order $8 \cdot 7^n$ must be solvable. (Hint: The action on the 7-Sylows gives a map to a symmetric group.)

Problem 6.4. Let $1 \to A \to B \to C \to 1$ be a short exact sequence of groups. Let B act on a set X in such a way that the action of A on X has exactly one orbit and trivial stabilizers. Show that the sequence $1 \to A \to B \to C \to 1$ is right split. (This problem does not involve Sylow's, but it will be useful soon.)

Problem 6.5. The aim of this problem is to prove the ***normalizer property*** of nilpotent groups: Let G be a nilpotent group and $H \subsetneq G$ a proper subgroup. Then $N_G(H) \supsetneq H$. (Of course, $N_G(H) \supseteq H$ is always true, so the surprise is the inequality.) As you would guess, the proof is by induction on $\#(G)$. Let Z be the center of G.

(1) Explain how to finish the proof if $Z \not\subseteq H$.
(2) Explain how to finish the proof if $Z \subseteq H$.

Problem 6.6. This problem follows up on Problems 5.6 and 5.7; you may use their results without proof. We define a polynomial $f(x)$ to be ***square-free*** if it is not divisible by $g(x)^2$ for any nonconstant polynomial $g(x)$.

(1) Let $K \subset L$ be fields of characteristic zero, and let $f(x) \in K[x]$. Show that f is square-free in $K[x]$ if and only if it is square-free in $L[x]$.
(2) Show that this result need not hold in nonzero characteristic.

Problem 6.7. Let R be a commutative, Noetherian[1] ring. Let S be a larger ring containing R and let $\theta \in S$. We say that θ is ***integral*** over R if θ obeys a polynomial relation of the form $\theta^n + \sum_{j=0}^{n-1} r_j \theta^j = 0$, with $r_j \in R$.

(1) Show θ is integral over R if and only if the ring $R[\theta]$ is finitely generated as an R-module.
(2) Show that the set of elements of S which are integral over R is a subring of S. This is called the ***integral closure*** of R in S.
(3) (**The rational root theorem**) Let R be a UFD, let $f_n x^n + \cdots + f_1 x + f_0$ be a polynomial with coefficients in R and assume f_0 and $f_n \neq 0$. Suppose that $\theta \in K$ is a root of f. Show that θ can be written in the form $\frac{a}{b}$ where $a | f_0$ and $b | f_n$.
(4) Let R be a UFD and let K be the fraction field of R. Show that $\theta \in K$ is integral over R if and only if $\theta \in R$.

[1]The results in this problem are true without the Noetherian hypothesis, but it isn't worth working hard enough to show that.