Definition. A group G is called ***simple*** if G has precisely two normal subgroups, G and $\{1\}$.

We remark that the trivial group is not simple, since it only has one normal subgroup.

Problem 6.1. Let G be simple and let H be any group. Show that, for every group homomorphism $\phi : G \to H$, either ϕ is injective or else ϕ is trivial.

Problem 6.2. Let p be a prime. Show that C_p is simple.

Problem 6.3. In this problem, we use a slick trick to check that A_5 is simple. The conjugacy classes of A_5 are as follows. (You may trust this; it will probably show up on homework eventually.)

representative element	e	(123)	$(12)(34)$	(12345)	(12354)
size of conjugacy class	1	20	15	12	12

(1) Show that any normal subgroup of A_5 must have size contained in the list

$$\left\{ \begin{matrix} 1,\ 1+12,\ 1+15,\ 1+20, \\ 1+12+12,\ 1+12+15,\ 1+12+20,\ 1+15+20, \\ 1+12+12+15,\ 1+12+12+20,\ 1+12+15+20, \\ 1+12+12+15+20 \end{matrix} \right\} = \{1,\ 13,\ 16,\ 21,\ 25,\ 28,\ 33,\ 36,\ 40,\ 45,\ 48,\ 60\}.$$

(2) Explain why the only possibilities in this list which can occur are 1 and 60.

Problem 6.4. Let G_1 and G_2 be simple groups and let N be a normal subgroup of $G_1 \times G_2$. Prove that one of the following cases must hold:

I: $N = \{1\}$
II: $N = G_1 \times \{1\}$
III: $N = \{1\} \times G_2$
IV: $N = G_1 \times G_2$ or
V: $G_1 \cong G_2$ and $N = \{(g, \phi(g)) : g \in G_1\}$ where $\phi : G_1 \to G_2$ is an isomorphism.

Hint: Think of ways to use N to make subgroups of G_1 and G_2.

Problem 6.5. Let G be a group and let N_1 and N_2 be distinct normal subgroups of G such that G/N_1 and G/N_2 are simple. Show that $G/(N_1 \cap N_2) \cong G/N_1 \times G/N_2$. Hint: What can the image of G in $G/N_1 \times G/N_2$ be?

After the prime cyclic groups C_p, the two most important families of simple groups are the alternating groups A_n (for $n \geq 5$), and the projective special linear groups $\mathrm{PSL}_n(F)$ (other than $\mathrm{PSL}_2(\mathbb{F}_2)$ and $\mathrm{PSL}_2(\mathbb{F}_3)$). Here, for any field F, the group $\mathrm{PSL}_n(F)$ is defined to be $\mathrm{SL}_n(F)/\,\mathrm{SL}_n(F) \cap Z$ where $\mathrm{SL}_n(F)$ is $n \times n$ matrices with determinant 1 and Z is matrices of the form $z\,\mathrm{Id}_n$ for $z \in F^{\times}$.

All proofs that these groups are simple are a bit lengthy; I have not decided to what extent we will prove this claim.