

PROBLEM SET 7: DUE WEDNESDAY, MARCH 18

Please see the course website for guidance on collaboration and formatting your problem sets. This problem set focuses on problems that will be useful as we begin the study of fields and their automorphisms.

Problem 7.1. Let K and L be fields and let $\phi : K \rightarrow L$ be a ring homomorphism (this includes the condition that $\phi(1) = 1$). Show that ϕ is injective.

Problem 7.2. Let A be a finite abelian group such that, for every integer d , A contains at most d solutions to $a^d = 1$. Show that A is cyclic. You may use the classification of finite abelian groups.

Problem 7.3. Let n be a positive integer. Let L be a field in which $n \neq 0$ and where the polynomial $x^n - 1$ factors into linear factors $x^n - 1 = \prod (x - \theta_j)$.

- (1) Show that the θ_j are distinct.
- (2) Show that the θ_j form a cyclic subgroup of L^\times . (Hint: The most efficient route is problem 7.2.) A generator of this group is called a **primitive n -th root of unity** in L .
- (3) Let σ be an automorphism of L . Show that there is some integer a , relatively prime to n , such that $\sigma(\theta) = \theta^a$ for each root θ of $x^n - 1$.

Problem 7.4. Let K be a field of characteristic p . Let a be an element of K such that the polynomial $x^p - x - a$ is irreducible over k , and let $L = K[t]/(t^p - t - a)K[t]$.

- (1) Show that the roots of the polynomial $x^p - x - a$ in L are $t, t + 1, t + 2, \dots, t + p - 1$.
- (2) Show that the group of automorphisms of L which fix K (pointwise) is cyclic of order p .

Problem 7.5. Let L/K be a field extension of degree 2 (meaning that L has dimension 2 as a K -vector space).

- (1) Show that there are b and $c \in K$ such that $L \cong K[x]/(x^2 + bx + c)K[x]$.
- (2) Suppose that K does not have characteristic 2. Show that $L \cong K[\sqrt{b^2 - 4c}]$.

Problem 7.6. (Reduced Row Echelon form) Let M be a $d \times n$ matrix with entries in a field k . Let $1 \leq j_1 < j_2 < \dots < j_d \leq n$. We will say that M is **in reduced row echelon form with pivots in columns** j_1, j_2, \dots, j_d if

$$M_{rj} = \begin{cases} 1 & j = j_r \\ 0 & j = j_s \text{ for } s \neq r \\ 0 & j < j_r \end{cases}$$

We say that M is **in reduced row echelon form** if it is in reduced row echelon for some set of columns j_1, j_2, \dots, j_d . The aim of this problem is to check that every d -dimensional subspace of k^n is the row span of exactly one $d \times n$ matrix in reduced row echelon form.

- (1) Let M be in reduced row echelon form with pivots in columns j_1, j_2, \dots, j_d and let $V \subseteq k^n$ be its row span. Show how to recover the j_1, j_2, \dots, j_d from the data of how V intersects various subspaces of k^n .
- (2) Let V be a d -dimensional subspace of k^n . Show that there is at most one matrix M in reduced row echelon form with row span V .
- (3) Let V be a d -dimensional subspace of k^n . Show that there is at least one matrix M in reduced row echelon form with row span V .

Problem 7.7. Let L be a field, let H be a group of automorphisms of L and let F be the elements of L fixed by H . Suppose that V is an L -vector subspace of L^n and that H takes V to itself. Show that V has a basis whose elements lie in F^n . (Hint: Look at Problem 7.6.)

Problem 7.8. Let K be a field of characteristic p . For $\theta \in K$, define $F(\theta) = \theta^p$ (this is called the Frobenius map).

- (1) Show that $F : K \rightarrow K$ is a field homomorphism.
- (2) Show that F is injective.
- (3) If K is finite, show that F is bijective.