7. Subnormal series, composition series and the Jordan-Holder theorem

Today, we are going to want the result of Problem 6.5, so we repeat it:

Problem 6.5 again: Let G be a group and let N_1 and N_2 be distinct normal subgroups of G such that G/N_1 and G/N_2 are simple. Show that $G/(N_1 \cap N_2) \cong G/N_1 \times G/N_2$. Hint: What can the image of G in $G/N_1 \times G/N_2$ be?

Definition. A ***subnormal series*** of a group G is a chain of subgroups $G = G_0 \triangleright G_1 \triangleright G_2 \triangleright G_3 \triangleright \cdots \triangleright G_N = \{e\}$ where G_{j+1} is normal in G_j. A ***composition series*** is a subnormal series where each subquotient G_j/G_{j+1} is simple.

Problem 7.1. Show that every finite group has a composition series.

Problem 7.2. Show that S_4 has a composition series with subquotients C_2, C_3, C_2 and C_2.

Problem 7.3. Show that $\mathrm{GL}_2(\mathbb{F}_7)$ has a composition series with subquotients C_2, C_3, $\mathrm{PSL}_2(\mathbb{F}_7)$ and C_2. You may assume that $\mathrm{PSL}_2(\mathbb{F}_7)$ is simple.

Problem 7.4. Let G be a group with a composition series and let N be a normal subgroup of G. Show that N and G/N have composition series.

The aim of this worksheet is to prove:

Theorem (Jordan-Holder)**.** Let G be a group with two composition series $G_0 \triangleright G_1 \triangleright \cdots \triangleright G_M = \{e\}$ and $H_0 \triangleright H_1 \triangleright \cdots \triangleright H_N = \{e\}$. Then $M = N$ and the list of subquotents $(G_0/G_1, G_1/G_2, \ldots, G_{M-1}/G_M)$ is a permutation of $(H_0/H_1, H_1/H_2, \ldots, H_{N-1}/H_N)$.

Our proof will be by induction on $\min(M, N)$.

Problem 7.5. Prove the base case, where $\min(M, N) = 0$.

Problem 7.6. Explain why we are done if $G_1 = H_1$.

Problem 7.7. Suppose that $G_1 \neq H_1$. Explain how to finish the proof in this case.

The Jordan-Holder theorem gives the basic strategy for studying groups: Understand the simple groups, and understand how they can be assembled into short exact sequences.