Problem Set 8: Due Wednesday, March 25

Problem 8.1. Let n be a positive integer and let K be a field in which $n \neq 0$. Let c be a nonzero element of K and let L be a splitting field of $x^n - c$.

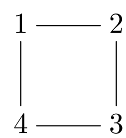
(1) Show that the polynomial $x^n - 1$ splits in L.

Let ζ generate the cyclic group of roots of $x^n - 1$ in L. Let γ denote one of the roots of $x^n - c$ in L.

(2) Show that $K(\zeta)$ is a splitting field for $x^n - 1$ and $\mathrm{Aut}(K(\zeta)/K)$ is isomorphic to a subgroup of the unit group $(\mathbb{Z}/n\mathbb{Z})^\times$.
(3) Let $\sigma \in \mathrm{Aut}(L/K)$. Show that there are integers $a \in (\mathbb{Z}/n\mathbb{Z})^\times$ and $b \in \mathbb{Z}/n\mathbb{Z}$ such that $\sigma(\zeta) = \zeta^a$ and $\sigma(\gamma) = \zeta^b \gamma$.
(4) Show that $\mathrm{Aut}(L/K)$ is isomorphic to a subgroup of $(\mathbb{Z}/n\mathbb{Z})^\times \ltimes \mathbb{Z}/n\mathbb{Z}$.

Problem 8.2. Let K have characteristic not 2. Let $f(x) = x^4 + bx^2 + c$ and let L be a splitting field of f. We assume that f is separable, and we number the roots of f so that $\theta_3 = -\theta_1$ and $\theta_4 = -\theta_2$.

(1) Show that $\mathrm{Aut}(L/K)$ is contained in the group of symmetries of the square shown below:



(2) Check that $(\theta_1\theta_2)^2 = c$ and $(\theta_1^2 - \theta_2^2)^2 = b^2 - 4c$.
(3) Show that $\mathrm{Aut}(L/K)$ in contained in the subgroup $\langle (13),\ (24) \rangle$ of S_4 if and only if $b^2 - 4c$ is square in K.
(4) Show that $\mathrm{Aut}(L/K)$ in contained in the subgroup $\langle (12)(34),\ (14)(23) \rangle$ of S_4 if and only if c is square in K.
(5) Show that $\mathrm{Aut}(L/K)$ in contained in the subgroup $\langle (1234) \rangle$ of S_4 if and only if $c(b^2 - 4c)$ is square in K.

Problem 8.3. Let L/K be a field extension of degree n. For $\theta \in L$, let m_θ be the map $x \mapsto \theta x$ from L to L. Let $f(x) = x^d + f_{d-1}x^{d-1} + \cdots + f_0$ be the minimal polynomial of θ over K.

(1) Show that m_θ is a K-linear map.
(2) Express the minimal polynomial and characteristic polynomial of m_θ in terms of $f(x)$, d and n.
(3) The ***trace*** $T_{L/K}(\theta)$ is defined to be the trace of the linear map m_θ. Express $T_{L/K}(\theta)$ in terms of the f_j, d and n.
(4) The ***norm*** $N_{L/K}(\theta)$ is defined to be the determinant of the linear map m_θ. Express $N_{L/K}(\theta)$ in terms of the f_j, d and n.

Problem 8.4. Let p be a prime and let $q = p^n$.

(1) Let k be a field of characteristic p. Show that the roots of $x^q = x$ in k form a subfield of k.
(2) Define $\mathbb{F}_q$ to be the splitting field of $x^q - x$ over $\mathbb{Z}/p\mathbb{Z}$. Show that $\mathbb{F}_q$ is a field with q elements.
(3) Let F be any field with q elements. Show that F is a splitting field for $x^q - x$ over $\mathbb{F}_p$, and conclude that $F \cong \mathbb{F}_q$.