8. Semidirect products

Let A be a group and let C be a group with a left action on A by a map $\phi : C \to \mathrm{Aut}(A)$. In other words, we require that $\phi(c)(a_1 * a_2) = \phi(c)(a_1) * \phi(c)(a_2)$ as well as the usual left action axiom that $\phi(c_1 c_2)(a) = \phi(c_1)\big(\phi(c_2)(a)\big)$.

Definition. With the above notation, the ***semidirect product*** $A \rtimes_\phi C$ is defined as the set of ordered pairs $(a, c) \in A \times C$ with multiplication $(a_1, c_1) * (a_2, c_2) = (a_1 * \phi(c_1)(a_2),\ c_1 * c_2)$.

The subscript ϕ is often omitted when it is clear from context.

Note that we use a left action of C on A to define $A \rtimes C$. Likewise, given a right action of C on A, we define $C \ltimes A$. This may seem odd, but I promise it is less confusing this way.

Problem 8.1. Verify that $A \rtimes_\phi C$ is a group.

Problem 8.2. In the above setting, show that:

(1) $\{(a, 1)\}$ is a normal subgroup of $A \rtimes_\phi C$, isomorphic to A.
(2) $\{(1, c)\}$ is a subgroup of $A \rtimes_\phi C$, isomorphic to C.
(3) $\{(a, 1)\} \cap \{(1, c)\} = \{(1, 1)\}$.
(4) Every element of $A \rtimes_\phi C$ can be written uniquely in the form $(a, 1)(1, c)$ for $a \in A$, $c \in C$.

Problem 8.3. Let G be a group with subgroups A and C such that, for $c \in C$, we have $cAc^{-1} = A$. When this condition holds, we say that C ***normalizes*** A.

(1) Show that $\{ac : a \in A,\ c \in C\}$ is a subgroup of G. We call this subgroup AC.
(2) Suppose, in addition that $A \cap C = \{1\}$. Show that[1] $AC \cong A \rtimes C$.
(3) Suppose that $A \cap C = \{1\}$ and both that A normalizes C and C normalizes A. Show that $AC \cong A \times C$.

The rest of the worksheet is examples.

Problem 8.4. Give two actions of C_2 on C_3 such that $S_3 \cong C_3 \rtimes C_2$ for one action and $C_6 \cong C_3 \rtimes C_2$ for the other.

Problem 8.5. Let p be prime. Show that $C_{p^2} \not\cong C_p \rtimes C_p$ for any action of C_p on C_p.

Let k be a field and let V be a k vector space; we'll write V_+ for V considered as an additive group. Let $\mathrm{GL}(V)$ be a group of invertible k-linear maps $V \to V$. Let $\mathrm{Aff}(V)$ be the group of maps $V \to V$ of the form $\vec{v} \mapsto a\vec{v} + \vec{b}$ for $a \in \mathrm{GL}(V)$ and $\vec{b} \in V$.

Problem 8.6. Show that $\mathrm{Aff}(V) \cong V_+ \rtimes \mathrm{GL}(V)$.

Problem 8.7. Let $\dim_k V = n$. Show that $\mathrm{Aff}(V)$ is isomorphic to the group of $(n+1) \times (n+1)$ matrices of the form

$$\begin{bmatrix} * & * & \cdots & * & * \\ * & * & \cdots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & \cdots & * & * \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}.$$

[1] It is possible that $G = AC$ and $A \cap C = \{1\}$, yet neither of A nor C normalizes each other. An example is $G = S_4$ with A the three element subgroup generated by (123) and C the eight element subgroup generated by (1234) and $(12)(34)$. In this case, we do not get to write G as a semidirect product.