9. ABELIANIZATION AND THE COMMUTATOR SUBGROUP

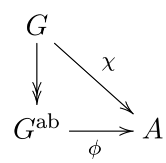
Definition. Let G be a group. The ***commutator subgroup*** is the group generated by all products $ghg^{-1}h^{-1}$ for g and $h \in G$. It is denoted $[G, G]$. The commutator subgroup is also called the ***derived subgroup***, and is sometimes also denoted G' or $D(G)$.

Problem 9.1. Show that $[G, G]$ is normal in G.

Problem 9.2. Show that $G/[G, G]$ is abelian.

Definition. The quotient $G/[G, G]$ is called the ***abelianization*** of G and denoted G^{ab}.

Problem 9.3. Prove the ***universal property of the abelianization***: If G is a group, A is an abelian group and $\chi : G \to A$ is a group homomorphism, then there is a unique homomorphism $\phi : G^{\mathrm{ab}} \to A$ such that the diagram below commutes:



Problem 9.4. Show that the commutator subgroup of S_n is A_n.

Problem 9.5. Show that the commutator subgroup of A_n is A_n for $n \geq 5$.

Remark. Way back in Problem 2.5, you showed that there are no nontrivial homomorphisms from A_5 to an abelian group. We can now state that result in a more sophisticated sounding way: The abelianization of A_5 is trivial.

Problem 9.6. Suppose that we have a short exact sequence $1 \to H \to G \to A \to 1$ where A is abelian. Show that $[G, G] \subseteq H$.

Problem 9.7. Many people believe that the every element in the commutator subgroup is of the form $ghg^{-1}h^{-1}$. This is need not be true. Let V be a vector space of dimension ≥ 4 over a field of characteristic $\neq 2$ and let G be the group whose underlying set is $V \times \bigwedge^2(V)$, with multiplication:

$$(v, \alpha) * (w, \beta) = (v + w, \alpha + \beta + v \wedge w).$$

You checked on the problem sets that G is a group.

(1) Show that $(0, \alpha)$ is a commutator if and only if α is of the form $v \wedge w$.
(2) Show that the commutator subgroup is all pairs $(0, \alpha)$ for $\alpha \in \bigwedge^2 V$.