## Problem Set 1 – due September 14

See the course website for policy on collaboration. **This problem set contains a trick question!**

**Notation:** The letter $k$ denotes an algebraically closed field. For $X \subseteq k^n$, we write $I(X)$ for the set of polynomials in $k[x_1, \dots, x_n]$ which vanish on all points of $X$. For $S \subseteq k[x_1, \dots, x_n]$, we write $Z(S)$ for the set of points $(a_1, \dots, a_n)$ on which all polynomials in $S$ vanish. Given a finitely generated $k$-algebra $A$, we write $\mathrm{MaxSpec}(A)$ for the set $\mathrm{Hom}_{k-\mathrm{alg}}(A, k)$.

1. We defined a subset $X$ of $k^n$ to be ***Zariski closed*** if $X$ is of the form $Z(S)$ for some $S \subseteq k[x_1, \dots, x_n]$ or, equivalently, if $X = Z(I(X))$. In this problem, we justify this terminology by verifying that the Zariski closed sets obey the axioms of a topology. Check that:
   (a) Any intersection of Zariski closed sets is Zariski closed.
   (b) Any finite union of Zariski closed sets is Zariski closed.
   (c) The sets $\emptyset$ and $k^n$ are Zariski closed.
2. Let $X$ be the curve $y^2 = x^3 + x^2$ in $\mathbb{A}^2$. We can parametrize $X$ as $t \mapsto (t^2 - 1, t(t^2 - 1))$.
   (a) What is the corresponding map $k[x, y]/\langle y^2 - x^3 - x^2 \rangle \to k[t]$?
   (b) Assume $k$ does not have characteristic 2. Show that this map has image $\{f \in k[t] : f(1) = f(-1)\}$.
3. Let $R = k[x, y]/(y^2 - x^3)$. Map $R \to k[t]$ by $x \mapsto t^2$ and $y \mapsto t^3$.
   (a) Show that $\mathrm{MaxSpec}\, k[t] \to \mathrm{MaxSpec}\, R$ is bijective.
   (b) Show that the inverse map is not regular.
4. Let $k[t] \times k[t]$ be the ring of ordered pairs $(f(t), g(t))$ of polynomials, with ring operations given by $(f_1, g_1) + (f_2, g_2) = (f_1 + f_2, g_1 + g_2)$ and $(f_1, g_1)(f_2, g_2) = (f_1 f_2, g_1 g_2)$.
   (a) Write $k[t] \times k[t]$ as $k[x, y]/I$ for some ideal $I$. (Hint: I suggest making $x$ correspond to $(t, t)$ and $y$ correspond to $(1, 0)$.)
   (b) Give a bijection between $\mathrm{MaxSpec}\, k[t] \times k[t]$ and $k \sqcup k$.
   (c) Map $k[t] \times k[t]$ to $k[t]$ by $(f, g) \mapsto f$. What is the corresponding map of MaxSpec's from $k$ to $k \sqcup k$?
   (d) Map $k[t]$ to $k[t] \times k[t]$ by $f \mapsto (f, 0)$. What is the corresponding map of MaxSpec's from $k \sqcup k$ to $k$?
5. Let $R$ be a commutative ring and $r$ an element of $R$. Show that $1 - rt$ is a unit of $R[t]$ if and only if $r$ is nilpotent in $R$. (***Nilpotent*** means that $r^N = 0$ for some positive integer $N$.)
6. Let $R$ be a commutative ring and let $I$ be an ideal of $R$. Define $\sqrt{I}$ (called the ***radical*** of $I$) to be $\{f \in R :\ \exists_{N \in \mathbb{Z}_{\geq 0}}\ f^N \in I\}$.
   (a) Show that $\sqrt{I}$ is an ideal.
   (b) Show that $\sqrt{\sqrt{I}} = \sqrt{I}$.
7. We work inside the ring $k[x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34}]$. We write $M$ for the matrix

$$M := \begin{pmatrix} 0 & x_{12} & x_{13} & x_{14} \\ -x_{12} & 0 & x_{23} & x_{24} \\ -x_{13} & -x_{23} & 0 & x_{34} \\ -x_{14} & -x_{24} & -x_{34} & 0 \end{pmatrix}.$$

   Let $I_4$ be the ideal generated by $\det M$ and let $I_3$ be the ideal generated by the determinants of the sixteen $3 \times 3$ submatrices of $M$.
   (a) One of the containments $I_3 \subset I_4$, $I_4 \subset I_3$ is true and the other is false; which is which?
   (b) Letting $I_a \subsetneq I_b$ denote the true relation, show that $\sqrt{I_a} = \sqrt{I_b} \subsetneq I_a \subsetneq I_b$.