Problem Set 10 – due November 30

Note extended due date, in light of Thanksgiving break.

See the course website for policy on collaboration.

1. In this problem, we will build an affine variety $X$ which is smooth in codimension 1 and a function in $\text{Frac}(X)$ which is regular in codimension 1, but not regular. Let $A$ be the subring of $k[x, y]$ generated by $x^4$, $x^3y$, $xy^3$ and $y^4$; we will abbreviate these monomials as $p$, $q$, $s$, $t$ respectively. Let $X = \text{MaxSpec } A$. Let $z$ be the point $p = q = s = t = 0$ of $X$. Let $r$ be the element $x^2y^2 = q^2/p = s^2/t \in \text{Frac}(A)$.
   
   (a) Show that $\dim X = 2$.
   
   (b) Show that the distinguished opens $\{p \neq 0\}$ and $\{t \neq 0\}$ cover $X \setminus \{z\}$.
   
   (c) Show that the distinguished opens $\{p \neq 0\}$ and $\{t \neq 0\}$ are smooth. So $X$ is smooth away from a single point, of codimension 2.
   
   (d) Show that $r$ is regular on $X \setminus \{z\}$.

2. Consider $\mathbb{P}^n$ with coordinates $(x_0 : x_1 : \cdots : x_n)$. Then $d \left( \frac{x_1}{x_0} \right) \wedge d \left( \frac{x_2}{x_0} \right) \wedge \cdots d \left( \frac{x_n}{x_0} \right)$ is a rational $n$-form on $\mathbb{P}^n$. Show that this $n$-form has no zeroes, is regular on $\{x_0 \neq 0\}$, and has a pole of order $n + 1$ along the hyperplane $x_0 = 0$.

3. Let $X$ be a smooth 1-dimensional variety, let $x_0$ be a point of $X$ and let $t$ be a regular function on $X$ generating the maximal ideal at $x_0$.
   
   (a) Let $\omega$ be a 1-form on $X \setminus \{x_0\}$. Show that $\omega$ can be uniquely written in the form
   
   $$\omega = a_{-N} \frac{dt}{t^N} + a_{-N+1} \frac{dt}{t^{N-1}} + \cdots + a_{-1} \frac{dt}{t} + \eta \quad (\star)$$
   
   where $\eta$ is a 1-form on $X$. Define $\text{res}_{t,x_0} \omega = a_{-1}$.
   
   In this problem we will show that $\text{res}_{t,x_0} \omega$ is independent of the choice of $t$. Let $u$ be another generator of $m_{x_0}$; we’ll show that $\text{res}_{t,x_0} (\omega) = \text{res}_{u,x_0} (\omega)$. To make the proof easier, assume that $\text{char}(k) = 0$, though this is also true in finite characteristic.
   
   (b) Let $g$ be a regular function on $X \setminus \{x_0\}$. Show that $\text{res}_{u,x_0} dg = 0$.
   
   (c) In the notation of $(\star)$, show that $\text{res}_{u,x_0} a_{-i} \frac{dt}{t^i} = 0$ for $i \geq 2$. Show that $\text{res}_{u,x_0} \eta = 0$. Show that $\text{res}_{u,x_0} dt/t = 1$. Conclude that $\text{res}_{u,x_0} \omega = \text{res}_{t,x_0} \omega$.
   
   From now on, we will just write $\text{res}_{x_0} (\omega)$.
   
   (d) Let $\omega$ be a rational 1-form on $\mathbb{P}^1$. Show that $\sum_{x_0 \in \mathbb{P}^1} \text{res}_{x_0} (\omega) = 0$. (Hint: Partial fractions!)

4. Recall that an integral domain $A$ is called **normal** if, for all $\theta \in \text{Frac}(A)$, if $\theta$ satisfies a monic polynomial with coefficients in $A$, then $\theta$ is in $A$. We call an affine variety $X$ **normal** if its ring of regular functions is normal. In this problem, we will see that normality is a local condition. Let $X$ be an irreducible affine variety with ring of regular functions $A$.
   
   (a) Suppose that $X$ has an open cover $X = \bigcup U_i$ where the $U_i$ are normal. Show that $X$ is normal. (Hint: Being a regular function is a local condition.)
   
   (b) Suppose that $A$ is a normal ring. Show that any localization $f^{-1}A$ is normal.
   
   (c) Show that $X$ is normal if and only if every affine open subset of $X$ is normal, if and only if $X$ has an open cover by normal affine varieties.