PROBLEM SET 10 – DUE NOVEMBER 30

Note extended due date, in light of Thanksgiving break.

See the course website for policy on collaboration.

1. In this problem, we will build an affine variety X which is smooth in codimension 1 and a function in $\mathrm{Frac}(X)$ which is regular in codimension 1, but not regular. Let A be the subring of $k[x,y]$ generated by x^4, x^3y, xy^3 and y^4; we will abbreviate these monomials as p, q, s, t respectively. Let $X = \mathrm{MaxSpec}\, A$. Let z be the point $p = q = s = t = 0$ of X. Let r be the element $x^2y^2 = q^2/p = s^2/t \in \mathrm{Frac}(A)$.
   (a) Show that $\dim X = 2$.
   (b) Show that the distinguished opens $\{p \neq 0\}$ and $\{t \neq 0\}$ cover $X \setminus \{z\}$.
   (c) Show that the distinguished opens $\{p \neq 0\}$ and $\{t \neq 0\}$ are smooth. So X is smooth away from a single point, of codimension 2.
   (d) Show that r is regular on $X \setminus \{z\}$.

2. Consider $\mathbb{P}^n$ with coordinates $(x_0 : x_1 : \cdots : x_n)$. Then $d\left(\frac{x_1}{x_0}\right) \wedge d\left(\frac{x_2}{x_0}\right) \wedge \cdots d\left(\frac{x_n}{x_0}\right)$ is a rational n-form on $\mathbb{P}^n$. Show that this n-form has no zeroes, is regular on $\{x_0 \neq 0\}$, and has a pole of order $n+1$ along the hyperplane $x_0 = 0$.

3. Let X be a smooth 1-dimensional variety, let x_0 be a point of X and let t be a regular function on X generating the maximal ideal at x_0.
   (a) Let ω be a 1-form on $X \setminus \{x_0\}$. Show that ω can be uniquely written in the form
   $$\omega = a_{-N}\frac{dt}{t^N} + a_{-N+1}\frac{dt}{t^{N-1}} + \cdots + a_{-1}\frac{dt}{t} + \eta \quad (*)$$
   where η is a 1-form on X. Define $\mathrm{res}_{t,x_0}\,\omega = a_{-1}$.
   In this problem we will show that $\mathrm{res}_{t,x_0}\,\omega$ is independent of the choice of t. Let u be another generator of $\mathfrak{m}_{x_0}$, we'll show that $\mathrm{res}_{t,x_0}(\omega) = \mathrm{res}_{u,x_0}(\omega)$. To make the proof easier, assume that $\mathrm{char}(k) = 0$, though this is also true in finite characteristic.
   (b) Let g be a regular function on $X \setminus \{x_0\}$. Show that $\mathrm{res}_{u,x_0}\, dg = 0$.
   (c) In the notation of $(*)$, show that $\mathrm{res}_{u,x_0}\, a_{-i}\frac{dt}{t^i} = 0$ for $i \geq 2$. Show that $\mathrm{res}_{u,x_0}\,\eta = 0$. Show that $\mathrm{res}_{u,x_0}\, dt/t = 1$. Conclude that $\mathrm{res}_{u,x_0}\,\omega = \mathrm{res}_{t,x_0}\,\omega$.
   From now on, we will just write $\mathrm{res}_{x_0}(\omega)$.
   (d) Let ω be a rational 1-form on $\mathbb{P}^1$. Show that $\sum_{x_0 \in \mathbb{P}^1} \mathrm{res}_{x_0}(\omega) = 0$. (Hint: Partial fractions!)

4. Recall that an integral domain A is called ***normal*** if, for all $\theta \in \mathrm{Frac}(A)$, if θ satisfies a monic polynomial with coefficients in A, then θ is in A. We call an affine variety X ***normal*** if its ring of regular functions is normal. In this problem, we will see that normality is a local condition. Let X be an irreducible affine variety with ring of regular functions A.
   (a) Suppose that X has an open cover $X = \bigcup U_i$ where the U_i are normal. Show that X is normal. (Hint: Being a regular function is a local condition.)
   (b) Suppose that A is a normal ring. Show that any localization $f^{-1}A$ is normal.
   (c) Show that X is normal if and only if every affine open subset of X is normal, if and only if X has an open cover by normal affine varieties.