Problem Set 11 – due December 7 (note extended due date)

See the course website for policy on collaboration.

1. Let X be a smooth irreducible variety. Show that X is normal. (Hint: Reduce to the affine case. Let $\theta \in \mathrm{Frac}(A)$ obey $\theta^n = \sum_{j=0}^{n-1} a_j \theta^j$. Show that, for each divisor D, we have $v_D(\theta) \geq 0$.)

2. This problem gives an example of a linear system with a base point. Consider $\mathbb{P}^2$ with coordinates $(x : y : z)$. Let H be the line $z = 0$. Then $H^0(\mathbb{P}^2, \mathcal{O}(2H))$ is the vector space with basis $\{1,\ x/z,\ y/z,\ (x/z)^2,\ (x/z)(y/z),\ (y/z)^2\}$. Let V be the subspace spanned by $\{x/z,\ y/z,\ (x/z)^2,\ (x/z)(y/z),\ (y/z)^2\}$.
   (a) Verify that the $(0 : 0 : 1)$ is a base point of V, and no other point is.
   Set $U = \mathbb{P}^2 \setminus \{(0 : 0 : 1)\}$. Let $\phi : U \to \mathbb{P}^4$ be the corresponding map. Let X be the closure of $\phi(U)$ in $\mathbb{P}^4$.
   (b) Verify that $X \setminus \phi(U)$ is isomorphic to $\mathbb{P}^1$. Conclude that ϕ cannot be extended to a regular map $\mathbb{P}^2 \to \mathbb{P}^4$.

3. We recall from the previous problem set the basic equations of the hyperelliptic curve. Assume the characteristic of k is not 2. Let $a_{2g+1}x^{2g+1} + a_{2g}x^{2g} + \cdots + a_2x^2 + a_1x$ be a squarefree polynomial in x with $a_1 a_{2g+1} \neq 0$. (Yes, the a_0 is meant to be missing.) Define the affine curves

$$\begin{aligned} X_0 &= \left\{ y_0^2 = a_{2g+1}x_0^{2g+1} + a_{2g}x_0^{2g} + \cdots + a_2x_0^2 + a_1x_0 \right\} \subset \mathbb{A}^2 \\ X_\infty &= \left\{ y_\infty^2 = a_{2g+1}x_\infty + a_{2g}x_\infty^2 + \cdots + a_2x_\infty^{2g} + a_1x_\infty^{2g+1} \right\} \subset \mathbb{A}^2 \end{aligned}.$$

   On previous problem sets, we showed that there is a smooth projective curve glued from two open sets isomorphic to X_0 and X_∞, with the gluing given by

$$x_0 = x_\infty^{-1} \qquad y_0 = y_\infty x_\infty^{-(g+1)}.$$

   We perform some more useful computations with this example.
   (a) Let p_∞ be the point $x_\infty = y_\infty = 0$ in X_∞ . Show that x_0 has a pole of order 2 at p_∞ and y_0 has a pole of order $2g + 1$.
   (b) Let n be a positive integer. Show that the vector space of functions on X which are regular on X_0 and have a pole of order $\leq n$ at p_∞ has dimension

$$\begin{cases} 1 + \lfloor n/2 \rfloor & n \leq 2g \\ n - g + 1 & n \geq 2g \end{cases}.$$

   (c) Let $\mathcal{O}(X_0)$, $\mathcal{O}(X_\infty)$ and $\mathcal{O}(X_0 \cap X_\infty)$ be the rings of regular functions on the corresponding sets. Show that the quotient

$$\mathcal{O}^1(X_0 \cap X_\infty) / \left(\mathcal{O}(X_0) + \mathcal{O}(X_\infty)\right)$$

   is a g-dimensional vector space.
   (d) Let $\Omega^1(X_0)$, $\Omega^1(X_\infty)$ and $\Omega^1(X_0 \cap X_\infty)$ be the modules of 1-forms on the corresponding sets. Show that the quotient

$$\Omega^1(X_0 \cap X_\infty) / \left(\Omega^1(X_0) + \Omega^1(X_\infty)\right)$$

   is a 1-dimensional vector space.

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4. In this problem, we will show that a normal curve is smooth. One can adapt this to prove the more general fact that the singular locus of a normal variety is codimension ≥ 2.

   Let X be a one dimensional, irreducible affine variety and let p be a point of X. Let A be the coordinate ring of X. We will show that, if $\dim T_p^* X > 1$, then A is not integrally closed in $\mathrm{Frac}(A)$. Let x_1 and x_2 be two functions vanishing at p and representing linearly independent classes in $T_p^* X$. Shrinking to a smaller neighborhood of X, we may (and do) assume that p is the only common zero of x_1 and x_2. So $(x_1 : x_2)$ is a regular map $X \setminus \{p\} \longrightarrow \mathbb{P}^1$. Let Y be the closure in $X \times \mathbb{P}^1$ of the graph of this map.

   (a) Let $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{GL}_2(k)$. Show that $\frac{ax_1+bx_2}{cx_2+dx_2}$ is not regular at p.

   (b) Show that there is a point of $\{p\} \times \mathbb{P}^1$ not in Y.

   After replacing $\left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right]$ by $\left[\begin{smallmatrix} ax_1+bx_2 \\ cx_1+dx_2 \end{smallmatrix}\right]$ for some $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{GL}_2(k)$, we may (and do) assume that the point $(p,\ (1:0))$ is not in Y. Having made this reduction, we put $\theta = \frac{x_1}{x_2}$. We know that θ is not regular at p.

   (c) Show that we can pass to a smaller neighborhood of p, so that $(X \times (1:0)) \cap Y = \emptyset$.

   From now on, we assume we have passed to such a smaller neighborhood. We now show that, having made all these reductions, θ is integral over A, so A is not integrally closed in $\mathrm{Frac}(A)$. Let I be the homogenous ideal of Y in $A[t_1, t_2]$ and let $B = A[t_1, t_2]/I$.

   (d) Show that there is an integer M such that t_1^M is divisible by t_2 in the ring B.

   (e) Show that there is a relation $t_1^M = \sum_{j=0}^{M-1} a_j t_1^j t_2^{M-j}$ in B, with $a_j \in A$, and deduce that $\theta^M = \sum_{j=0}^{M-1} a_j \theta^j$ in $\mathrm{Frac}(A)$.