Problem Set 2 – due September 21

See the course website for policy on collaboration.

Notation Throughout this problem set, k denotes an algebraically closed field.

1. Let A be an $n \times n$ matrix with entries in k. Let R be the commutative ring $k[A]$. What is the relationship between the eigenvalues of A (also called the ***spectrum*** of A) and $\mathrm{MaxSpec}\, R$?
2. Prove the Zariski topology on k^2 is **not** the product topology of the Zariski topology on k with itself.
3. Let R be a finitely generated k-algebra. Choose generators $x_1, x_2, \ldots, x_n$ and write $R = k[x_1, \ldots, x_n]/I$. So we have a natural bijection $\mathrm{MaxSpec}(R) \longleftrightarrow Z(I)$. Show that $Y \subseteq X$ is closed in the Zariski topology if and only if there is an ideal $J \subseteq R$ such that Y corresponds to the set of maximal ideals containing J.
4. Describe the images of the following maps. Are they open? Closed?
   (a) Map k^2 to k^2 by $(x, y) \mapsto (x, xy)$.
   (b) Let $SL_2 = \{\left(\begin{smallmatrix} w & x \\ y & z \end{smallmatrix}\right) : wz - xy = 1\}$. Map SL_2 to k^2 by $\left(\begin{smallmatrix} w & x \\ y & z \end{smallmatrix}\right) \mapsto (w, x)$.

The remainder of this problem set reminds us of the basic properties of Noetherian rings:

Let R be a commutative ring. We define the following properties of R, which we will then show are all equivalent. If R has any (and hence all) of these properties, we define R to be ***noetherian***.

$1(a)$ For any chain $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$ of ideals in R, we have $I_r = I_{r+1}$ for all sufficiently large r.
$1(b)$ Every ideal I of R is finitely generated.
$2(a)$ For any $n \geq 0$ and any chain $M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots$ of submodules of R^n, we have $M_r = M_{r+1}$ for all sufficiently large r.
$2(b)$ For any $n \geq 0$, every submodule M of R^n is finitely generated.
$3(a)$ For any finitely generated R-module S and any chain $M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots$ of submodules of S, we have $M_r = M_{r+1}$ for all sufficiently large r.
$3(b)$ For any finitely generated R-module S, every submodule M of S is finitely generated.

5. For your favorite choice of $\# \in \{1, 2, 3\}$, show that $\#(a)$ and $\#(b)$ are equivalent.
6. For your favorite choice of $* \in \{a, b\}$, show that:
   (a) $1(*) \implies 2(*)$.
   (b) $2(*) \implies 3(*)$.
   (c) $3(*) \implies 1(*)$.

   We remark that $1(b)$ is obvious for fields, but $2(b)$ is the first significant theorem in a linear algebra course, so you should expect to have to do some work.
7. Show that a quotient of a noetherian ring is noetherian. Hint: The $3(*)$ properties are your friend.
8. We will now prove the **Hilbert basis theorem**: If A is noetherian, then $A[t]$ is noetherian. Hence, by induction on n, $k[t_1, t_2, \ldots, t_n]$ is noetherian. Applying the previous problem, this shows any finitely generated k-algebra is noetherian.

   Let I be an ideal of $A[t]$. We will be proving Property 1(b), that I is finitely generated. Define I_d to be the set of $g \in A$ such that there is an element of I of the form $gt^d + f_{d-1}t^{d-1} + \cdots f_1 t + f_0$.
   (a) Show that I_d is an ideal of A.
   (b) Show that $I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots$.
   Using property $1(a)$ of A, there there is some ideal I_∞ of A so that $I_r = I_{r+1} = \cdots = I_\infty$. Using property $1(b)$ of A, take a finite list of generators $g_1, g_2, \ldots, g_k$ of I_∞. For each g_i, choose $f_i \in I$ of the form $g_i t^r +$ lower order terms.
   (c) Show that $I \cap A \cdot \{1, t, t^2, \ldots, t^{r-1}\}$ is finitely generated as an A-module.
   Let $h_1, h_2, \ldots, h_\ell$ be a list of generators for $I \cap A \cdot \{1, t, t^2, \ldots, t^{r-1}\}$.
   (d) Show that $f_1, f_2, \ldots, f_k, h_1, h_2, \ldots, h_\ell$ generate I as an $A[t]$ module.