Problem Set 3 – due September 26

See the course website for policy on collaboration.

Notation The symbol k denotes an algebraically closed field. Let X be a Zariski closed subset of k^m, let $\Omega \subseteq X$ be Zariski open and let f be a function $\Omega \to k$. For $x \in \Omega$ we say that f is ***regular at*** x if there is a Zariski open neighborhood U of x and regular functions g and h on X such that $h|_U \neq 0$ and $f|_U = (g/h)|_U$. We say that f is regular on Ω if it is regular at every point of Ω.

1. (This is the converse of a result proved in class.) Let R be a commutative ring and let I be an ideal of $R[y]$. Suppose that $R[y]/I$ is finitely generated as an R-module. Show that y obeys a monic polynomial $y^N + g_{N-1}y^{N-1} + \cdots + g_0$ in the quotient $R[y]/I$.
2. Let X be an affine variety. Show that the Zariski topology on X is compact.
3. Let k have characteristic not equal to 2. Let

$$O_2 = \left\{ \begin{pmatrix} w & x \\ y & z \end{pmatrix} : \begin{pmatrix} w & x \\ y & z \end{pmatrix} \begin{pmatrix} w & y \\ x & z \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

   (a) Show that O_2 is disconnected, explicitly writing it disjoint union of two closed components.
   (b) Find an idempotent in $\mathcal{O}_{O_2}$ other than 0 and 1.
4. Let X be $\{(w,x,y,z) : \ wy = x^2, \ xz = y^2\}$.
   (a) Show that X is not irreducible, explicitly writing $X = X_1 \cup X_2$ for two Zariski closed sets X_1 and X_2 with X_1, $X_2 \neq X$.
   (b) Show that the X_1 and X_2 you found are irreducible.
5. Decompose X into irreducible components where

$$X = \left\{ \begin{pmatrix} u & v & w \\ x & y & z \end{pmatrix} : \det \begin{pmatrix} u & v \\ x & y \end{pmatrix} = \begin{pmatrix} v & w \\ y & z \end{pmatrix} = 0 \right\}$$

6. Let X be an affine variety. Show that the units in the ring $\mathcal{O}_X$ are precisely the regular functions $f : X \to k$ which are nowhere 0.
7. Let $X \subseteq k^m$ be Zariski closed, and let $f : X \to k^n$ be a regular map. Define the graph of f, written $\Gamma(f)$, to be the set $\{(x,y) \in X \times k^n : f(x) = y\}$.
   (a) Show that $\Gamma(f)$ is Zariski closed in $k^m \times k^n$.
   (b) Show that $\Gamma(f) \cong X$. That is to say, show that there are regular maps $\Gamma(f) \to X$ and $X \to \Gamma(f)$ which are mutually inverse.
8. Let $X = \mathbb{A}^2$, with coordinate functions x and y. Let $\Omega = X \setminus \{(0,0)\}$. Show that any regular function on Ω extends to a regular function on X.
9. Let $X = \{(w,x,y,z) : wy = wz = xy = xz = 0\}$. Let $\Omega = X \setminus \{(0,0,0,0)\}$. This problem gives an example of a regular function on Ω which cannot be represented as g/h for a single polynomial h which is nonzero everywhere on Ω.
   (a) Decompose X into irreducible components. You should find two, call them X_0 and X_1.
   (b) Show that $X_0 \cap \Omega$ and $X_1 \cap \Omega$ are disjoint.
   (c) Let $f : \Omega \to k$ be 0 on X_0 and 1 on X_1. Show that f is regular on Ω.
   (d) Show that f cannot be written in the form g/h with g and $h \in \mathcal{O}_X$ and h nowhere vanishing on Ω.