PROBLEM SET 6 – DUE WEDNESDAY, ~~OCTOBER 24~~ OCTOBER 31

See the course website for policy on collaboration.

1. Let X be a quasiprojective variety and let U and V be open affine subvarieties. Show that $U \cap V$ is affine. (Hint: Embed $U \cap V$ into $U \times V$.)

2. (a) Let $X = \operatorname{MaxSpec} A$ be an irreducible affine variety (so that A is an integral domain) and let $U = \operatorname{MaxSpec} B$ be a nonempty open affine subvariety. Show that $\operatorname{Frac} A \cong \operatorname{Frac} B$.
   (b) Now let X be an irreducible quasiprojective affine variety and let $U = \operatorname{MaxSpec} A$ and $V = \operatorname{MaxSpec} B$ be nonempty open affine subvarieties. Show that $\operatorname{Frac} A \cong \operatorname{Frac} B$.

3. Let $\phi : \mathbb{A}^1 \to \mathbb{A}^n$ be a regular map. Show $\phi(\mathbb{A}^1)$ is closed. (Hint: Show that ϕ is either finite or constant.)

4. The **blow up of** $\mathbb{A}^n$, denoted $B\ell_0\mathbb{A}^n$, is defined to be the subset of $\mathbb{A}^n \times \mathbb{P}^{n-1}$ given by the equations $x_i y_j - x_j y_i = 0$, where $(x_1, \ldots, x_n)$ are the coordinates on $\mathbb{A}^n$ and $(y_1 : \cdots : y_n)$.
   (a) Describe the fibers of the projection $B\ell_0\mathbb{A}^n \to \mathbb{A}^n$.
   (b) Describe the fibers of the projection $B\ell_0\mathbb{A}^n \to \mathbb{P}^{n-1}$.
   (c) Let $Z \subset \mathbb{A}^2$ be $\{(x_1, x_2) : x_1 x_2 (x_1 - x_2) = 0\}$. Describe the subset of $B\ell_0$ lying above Z.

5. Compute the dimensions of the following varieties:
   (a) $\{(w, x, y, z) : wz - xy = 0\}$
   (b) $\{(w, x, y, z) : wz - xy = w = x = 0\}$
   (c) $\{(w, x, y, z) : wy = x^2,\ xz = y^2,\ wz = xy\}$.

6. Let $\operatorname{Mat}_{a \times b}$ denote $\mathbb{A}^{ab}$, thought of as the space of $a \times b$ matrices. Let $0 \leq r \leq m, n$. Let μ be the map $\operatorname{Mat}_{m \times r} \times \operatorname{Mat}_{n \times r} \to \operatorname{Mat}_{m \times n}$ defined by $\mu(A, B) = \mu(AB)$.
   (a) Show that the image of μ is the set of matrices of rank $\leq r$.
   (b) Show that the fiber of μ above a matrix of rank r has dimension r^2.
   (c) Compute the dimension of the set of matrices of rank $\leq r$.

7. (This is a lemma which I expect to want sometime in the last week of October.) Let V be a finite dimensional vector space, ω an element of $\bigwedge^k V$, and $v_1, v_2, \ldots, v_r$ linearly independent vectors in V. Suppose that $v_i \wedge \omega = 0$, for $1 \leq i \leq r$. Show that we can write $\omega = v_1 \wedge v_2 \wedge \cdots \wedge v_r \wedge \eta$ for some $\eta \in \bigwedge^{k-r} V$.