PROBLEM SET 7 – DUE WEDNESDAY, ~~OCTOBER 31~~ NOVEMBER 7

See the course website for policy on collaboration.

1. Let X be a Zariski closed subset of $\mathbb{P}^n$ with corresponding homogenous ideal $I \subseteq k[x_0, \ldots, x_n]$. The **Hilbert function** is defined to be $h(d) := \dim_k k[x_0, \ldots, x_n]_d / I_d$. We'll start this topic in class probably on Monday, but you can get started with the following examples earlier:
   (a) Compute the Hilbert function of $\{(1:0:0), (0:1:0), (0:0:1)\}$ in $\mathbb{P}^2$.
   (b) Compute the Hilbert function of $\{(1:0:0), (1:1:0), (0:1:0)\}$ in $\mathbb{P}^2$.
   (c) Let f and g be relatively prime polynomials in $k[x_0, x_1, x_2, x_3]$, of degrees a and b, with $\langle f, g \rangle$ radical. Compute the Hilbert function of $Z(f, g)$.

   Hint for the next two problems: Recall the theorems on the dimension of fibers.

2. Let V_d be the vector space of degree d polynomials in (x, y, z). For any n points $p_1, p_2, \ldots, p_n$ in $\mathbb{P}^2$, let $V_d(p_1, \ldots, p_n)$ be the subspace of polynomials vanishing at $p_1, p_2, \ldots, p_n$. Show that there is a nonempty Zariski open subset Ω of $(\mathbb{P}^2)^n$ such that, for $(p_1, \ldots, p_n) \in \Omega$, we have

$$\dim V_d(p_1, \ldots, p_n) = \max\left(\dim V_d - n, 0\right).$$

3. Let V be a vector space of dimension n and let $\bigwedge^d V$ be the d-th wedge power. For $\omega \in \bigwedge^d V$, consider the map $(\omega \wedge \quad) : V \to \bigwedge^{d+1}(V)$.
   (a) For $\omega \neq 0$, show that $\dim \operatorname{Ker}(\omega \wedge \quad) \leq d$.
   (b) Show that the set of $[\omega]$ in $\mathbb{P}(\bigwedge^d V)$ for which $\dim \operatorname{Ker}(\omega \wedge \quad) = d$ is Zariski closed.

4. Let k not have characteristic 2 or 3. Let A be the ring $k[x, y]/(y^2 - x^3 - x)$. Let D be the A-module generated by symbols dx and dy modulo the relation

$$2y dy = (3x^2 + 1) dx.$$

   Show that D is a free A-module. Give an explicit generator ω and formulas for dx and dy as multiples of ω. (Hint: The proof of Theorem 4 in Shavarevich I.3.2 may be inspirational.)

5. Let X be a closed subvariety of $\mathbb{A}^n$ with ideal I. Let $k[x_1, \ldots, x_n]_{\leq t}$ denote the set of polynomials of degree $\leq t$. Suppose that X has a surjective Noether normalization $\pi : X \to \mathbb{A}^d$ of degree δ. (This last means that $A \otimes \operatorname{Frac} \mathbb{A}^d$ is dimension δ as a $\operatorname{Frac} \mathbb{A}^d$ vector space.) Show that

$$\dim k[x_1, \ldots, x_n]_{\leq t} / (I \cap k[x_1, \ldots, x_n]_{\leq t}) = \tfrac{\delta}{d!} t^d + O(t^{d-1}).$$