Problem Set 8 – due Wednesday, November 14

See the course website for policy on collaboration.

1. Let A be a commutative ring and $\mathfrak{m}$ a maximal ideal. We have previously proved Nakayama's lemma in the form: Let M be a finitely generated A-module.If $M/\mathfrak{m}M = 0$, then there is some $f \equiv 1 \bmod \mathfrak{m}$ such that $f^{-1}M = 0$. Prove the following variants:
   (a) Let M_1 and M_2 be finitely generated A-modules and $M_1 \to M_2$ a map of A-modules. If the induced map $M_1/\mathfrak{m}M_1 \to M_2/\mathfrak{m}M_2$ is surjective, then there is some $f \equiv 1 \bmod \mathfrak{m}$ such that $f^{-1}M_1 \to f^{-1}M_2$ is surjective.
   (b) Let M be a finitely generated A-module and let $v_1, v_2, \dots, v_r \in M$ and let $\overline{v}_j$ be the class of v_j in $M/\mathfrak{m}M$. Suppose that $\overline{v}_1, \dots, \overline{v}_r$ span $M/\mathfrak{m}M$ as an $A/\mathfrak{m}$ vector space. Show that there is some $f \equiv 1 \bmod \mathfrak{m}$ such that $v_1, \dots, v_r$ generate $f^{-1}M$ as an $f^{-1}A$ module.
2. This question takes place on $\mathbb{P}^1$ with homogenous coordinates $[z_1 : z_2]$. Let $x = z_1/z_2$.

   Show that the space of globally regular vector fields on $\mathbb{P}^1$ is three dimensional, spanned by the vector fields whose values on the open set $\{z_2 \neq 0\}$ are $\frac{\partial}{\partial x}$, $x\frac{\partial}{\partial x}$, $x^2\frac{\partial}{\partial x}$.
3. In this question, take $\mathrm{char}(k) \neq 2$. Let $a(x) = a_{2g+1}x^{2g+1} + a_{2g}x^{2g} + \cdots + a_2x^2 + a_1x$ be a squarefree polynomial in x with $a_1a_{2g+1} \neq 0$. (Yes, the a_0 is meant to be missing.) Let X be the affine curve $y^2 = a(x)$ in $\mathbb{A}^2$, and let A be its coordinate ring.
   (a) Show that X is smooth.
   (b) Show that there is a regular 1-form ω_1 on X with $\omega = \frac{dx}{2y} = \frac{dy}{a'(x)}$ wherever the second and third expression are defined. Show that Ω^1_A is the free A-module with basis ω.

   We now restrict to the case $g = 1$. (Higher g will return on later problem sets).
   (c) Let E be the curve $z_2^2z_3 = a_3z_1^3 + a_2z_1^2z_3 + a_1z_1z_3^2$. Show that $E \cap \{z_3 \neq 0\}$ is isomorphic to X. Show that ω extends to a regular 1-form on all of E.
4. Let $X = Z(x_1x_2) \subset \mathbb{A}^2$. Let A be the coordinate ring, $k[x_1, x_2]/\langle x_1x_2\rangle$.
   (a) Show that the 1-form x_1dx_2 vanishes everywhere on TX.
   (b) Show that there is a well-defined A-linear map $\Omega^1_A \to k[x_1, x_2]/\langle x_1^2, x_2^2\rangle$ sending $dx_1 \mapsto x_1$ and $dx_2 \mapsto -x_2$. Deduce that x_1dx_2 is not zero in Ω^1_A.
5. This problem introduces the Grassmannian, an object which I sadly did not have the chance to get to in class. Let $0 < d < n$. The Grassmannian $G(d, n)$ will be a variety whose points are in natural bijection with the set of d-planes in k^n. Let $U \subset \mathbb{A}^{dn}$ be the set of $n \times d$ matrices of rank d. Let $\phi : U \to \mathbb{P}(\bigwedge^d k^n)$ be the map which sends the matrix with columns $v_1, v_2, \dots, v_d$ to $[v_1 \wedge v_2 \cdots \wedge v_d]$.
   (a) Explain why Problem 7 on Problem Set 6, together with Problem 3 on Problem Set 7, show that the image of ϕ is closed in $\mathbb{P}(\bigwedge^d k^n)$. Calling that image $G(d.n)$, explain why $G(d, n)$ is in bijection with the set of d-planes in n-space.
   (b) Let $e_1, \dots, e_n$ be the standard basis of k^n, so $\{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_d} : 1 \leq i_1 < i_2 < \cdots < i_d \leq n\}$ is a basis of $\bigwedge^d k^n$. Let $V \subset \mathbb{P}(\bigwedge^d k^n)$ be the open set where the coefficient of $e_1 \wedge e_2 \wedge \cdots \wedge e_d$ is nonzero. Show that each point of $G(d, n) \cap V$ is of the form $\phi\left(\begin{smallmatrix} \mathrm{Id}_d \\ M \end{smallmatrix}\right)$ for a unique $(n-d) \times d$ matrix M.
   (c) Show that $G(d, n) \cap V \cong \mathbb{A}^{d(n-d)}$.