Problem Set 9 – due November 21

See the course website for policy on collaboration. This problem set is due the day before Thanksgiving. If you won't be here, please turn it in electronically.

1. Let $\mathrm{char}(k) \neq 3$. Let F be the curve $x^3 + y^3 + z^3 = 0$ in $\mathbb{P}^2$, and let ω be a primitive cube root of 1. In this problem, we'll use F to practice computing differentials of maps.
   (a) Let ϕ be the map $(x : y : z) \mapsto (x : y)$ from F to $\mathbb{P}^1$. Find the points of F where $\phi_* : T_{(x:y:z)}F \to T_{(x:y)}\mathbb{P}^1$ is 0.
   Define $\psi : F \to F$ by
   $$\psi(x : y : z) = \left(-3xyz : x^3 + \omega y^3 + \omega^2 z^3 : x^3 + \omega^2 y^3 + \omega z^3\right).$$
   This is a so-called 3-isogeny of the genus 1 curve F.
   (b) Show that ψ_* is nonzero on every tangent space.
   (c) The point $(0 : 1 : -1)$ is fixed by ψ. Compute the scalar by which ϕ_* multiplies $T_{(0:1:-1)}F$.

2. Let $\mathrm{char}(k) = 0$. Let $f_1, f_2, \ldots, f_r \in k[x, y]$ be polynomials with no common zeroes (in other words, $\bigcap_i Z(f_i) = \emptyset$). Let $X \subset \mathbb{A}^2 \times \mathbb{A}^r$ be the set of $\{(x, y, c_1, c_2, \ldots, c_r) : \sum c_j f_j(x, y) = 0\}$.
   (a) Show that X is smooth.
   (b) Show that, for generic $c_1, \ldots, c_r$, the curve $Z\left(\sum c_j f_j(x, y)\right)$ is smooth.

3. In this problem, we will prove Bertini's theorem without using the characteristic 0 hypothesis. That is to say, let V be an n-dimensional k vector space and let X be a smooth d-dimensional subvariety of $\mathbb{P}(V)$. For $y \in \mathbb{P}(V^\vee)$, let H_y be the projective hyperplane $\{\langle y, \ \rangle = 0\}$ in $\mathbb{P}(V)$. We will show that, for generic $y \in \mathbb{P}(V)$, the intersection $X \cap H_y$ is smooth.
   (a) Define $Z \subset \mathbb{P}(V) \times \mathbb{P}(V^\vee)$ to be the set of pairs (x, y) obeying the conditions: $x \in X$, $x \in H_y$ and $T_x X \subseteq T_x H_y$. Show that Z has dimension $n - 2$.
   (b) Deduce that there is some $y \in \mathbb{P}(V^\vee)$ such that, at every point of $X \cap H_y$, the planes $T_x X$ and $T_x H_y$ are transverse.

4. In this problem, we check that the Kleiman-Bertini theorem is not true in characteristic p, so let $\mathrm{char}(k) = p$. Let $Z = \mathbb{P}^1 \times \mathbb{P}^1$, which has a transitive action of $G := \mathrm{PGL}_2 \times \mathrm{PGL}_2$. Take $X = \mathbb{P}^1 \times \{\text{point}\}$. Give an example of a smooth variety $Y \subset \mathbb{P}^1 \times \mathbb{P}^1$ such that gX and Y do **not** meet transversely for any $g \in G$.

5. In this problem, we continue the discussion of hyperelliptic curves from Problem 3 on the last set. Assume the characteristic of k is not 2. Let $a_{2g+1}x^{2g+1} + a_{2g}x^{2g} + \cdots + a_2 x^2 + a_1 x$ be a squarefree polynomial in x with $a_1 a_{2g+1} \neq 0$. (Yes, the a_0 is meant to be missing.) Define the affine curves
   $$\begin{aligned} X_0 &= \left\{ y_0^2 = a_{2g+1}x_0^{2g+1} + a_{2g}x_0^{2g} + \cdots + a_2 x_0^2 + a_1 x_0 \right\} \subset \mathbb{A}^2 \\ X_\infty &= \left\{ y_\infty^2 = a_{2g+1}x_\infty + a_{2g}x_\infty^2 + \cdots + a_2 x_\infty^{2g} + a_1 x_\infty^{2g+1} \right\} \subset \mathbb{A}^2 \end{aligned}.$$
   From the previous problem, these are smooth curves and the modules of differentials on these curves are free, with generators:
   $$\omega_0 = \frac{dx}{2y_0} = \frac{dy}{\sum_j j a_j\, x_0^{j-1}} \qquad \omega_\infty = \frac{dx}{2y_\infty} = \frac{dy}{\sum_j j a_{2g+2-j}\, x_\infty^{j-1}}.$$

Please turn over!

This problem will study a curve X which is covered by two open sets, isomorphic to X_0 and X_∞, and glued by

$$x_0 = x_\infty^{-1} \qquad y_0 = y_\infty x_\infty^{-(g+1)}.$$

Since we can't glue varieties abstractly, we start by building X as an explicit subvariety of $\mathbb{P}^{g+2}$. Map X_0 and X_∞ into $\mathbb{P}^{g+2}$ by

$$\iota_0(x_0, y_0) = (1 : x_0 : \cdots : x_0^g : x_0^{g+1} : y_0) \qquad \iota_\infty(x_\infty, y_\infty) = (x_\infty^{g+1} : x_0^g : \cdots : x_0 : 1 : y_0).$$

We write $(z_0 : z_1 : \cdots : z_g : z_{g+1} : z_{g+2})$ for the homogenous coordinates on $\mathbb{P}^{g+2}$.

(a) Show that ι_0 is injective, that $\iota_0(X_0)$ is closed in the affine space $\{z_0 \neq 0\}$, and that ι_0 is an isomorphism onto its image.

After doing this computation, you may assume without proof that the corresponding statements hold for ι_∞.

(b) Show that $\iota_0\left(X_0 \setminus (0,0)\right) = \iota_\infty\left(X_\infty \setminus (0,0)\right)$, and they are glued by the formulas above.

(c) Put $X = \iota_0(X_0) \cup \iota_\infty(X_\infty)$. Show that X is closed in $\mathbb{P}^{g+2}$. (Hint: Consider each affine chart separately.)

We now may speak of the curve X.

(d) Show that ω_0 extends to a global 1-form on X. Give a formula for ω_0 on X_∞ in terms of ω_∞, x_∞ and y_∞.

(e) Show that the vector space of 1-forms on X has dimension g.