Nakayama's Lemma – optional problem set

Problem 1 Let R be a commutative ring and M an A-module. Let A be an $n \times n$ matrix with entries in R and let v be an n-component vector with entries in M. Suppose that $Av = 0$ (this is n linear equations in the module M). Show that $(\det A)v = 0$. (Again, this is n linear equations taking place in M.)

Problem 2 (Nakayama's Lemma) Let R be a commutative ring, let I be an ideal and let M be a finite generated R module. Suppose that $IM = M$. Show that there is some element $f \in R$ with $R/I \neq 0$ such that $fM = 0$.

Hint: Let $v_1, v_2, \ldots, v_n$ generate M. By hypothesis, we have $v_i = \sum_j b_{ij} v_j$ for some $n \times n$ matrix B. Now figure out how to apply Problem 1.

Problem 3 Here are some of the forms in which I usually apply Nakayama's lemma. In all of these problems, R is a commutative ring, $\mathfrak{m}$ is a maximal ideal of R and $R/\mathfrak{m} = k$.

(a) Let M be a finitely generated R-module and suppose that $M/\mathfrak{m}M = 0$. Then there is an element $f \in R$, whose image in k is nonzero, so that $f^{-1}M$ (considered as a module for $f^{-1}R$) is zero.

(b) Let N be a finitely generated R-module and $u_1, u_2, \ldots, u_n$ elements of N. Let the images of the u_i span $N/\mathfrak{m}N$ (as a k-vector space). Then there is an element $f \in R$, whose image in k is nonzero, such that the u_i generate $f^{-1}N$ as an $f^{-1}R$-module.

(c) Let P and Q be finitely generated R-modules and $\phi : P \to Q$ a map of R-modules. Suppose that the induced map $P/\mathfrak{m}P \to Q/\mathfrak{m}Q$ is a surjection (of k-vector spaces). Then there is an element $f \in R$, whose image in k is nonzero, such that the map $f^{-1}P \to f^{-1}Q$ is a surjection (of $f^{-1}R$-modules).

(d) Let N be a free R-module of finite rank and let $u_1, u_2, \ldots, u_n$ elements of N. Let the images of the u_i be a basis for $N/\mathfrak{m}N$ (as a k-vector space). Then n is the rank of N and there is an element $f \in R$, whose image in k is nonzero, such that the u_i are a basis for the free $f^{-1}R$-module $f^{-1}N$.

Problem 4 Let $R = k[t]$, let M be $k[t, t^{-1}]$ considered as an R-module and let $I = tR$.

(a) Show that $M = IM$, but there is no $f \in R \setminus I$ for which $fM = 0$.

(b) Use this example to build counter-examples to 3.(a), (b) and (c) when the finite generation hypotheses are removed.

Problem 5 Give an example of the following situation: R is a commutative ring, $\mathfrak{m}$ a maximal ideal, with $k := R/\mathfrak{m}$ and M is a finitely generated R-module, such that $\mathfrak{m}M = M$ but $M \neq 0$.

Problem 6 (a) Give an example of the following situation: R is a commutative ring, $\mathfrak{m}$ a maximal ideal, with $k := R/\mathfrak{m}$, N is a finitely generated R-module and $u_1, \ldots, u_n$ are elements of N, such that the u_i are a basis for the k-vector space $N/\mathfrak{m}N$, but the u_i obey nontrivial linear relations in R. (Nontrivial means a relation $\sum c_i u_i = 0$ with the c_i not all 0.)

(b) Give an example as above where the u_i obey nontrivial linear relations in $f^{-1}R$ for every $f \in R \setminus \mathfrak{m}$.