OPTIONAL PROBLEM SET – THE NOETHERIAN PROPERTY AND THE HILBERT BASIS THEOREM

This problem set is optional for you: If you are feeling rusty on Noetherianness, and the Hilbert basis theorem, work through whatever parts aren't obvious to you. I'll be glad to grade it for anyone who would like me to.

Let R be a commutative ring. We define the following nine properties of R, which we will then show are all equivalent. If R has any (and hence all) of these properties, we define R to be ***noetherian***.

- $1(a)$ For any chain $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$ of ideals in R, we have $I_r = I_{r+1}$ for all sufficiently large r.
- $1(b)$ Every ideal I of R is finitely generated.
- $1(c)$ For any set $\mathcal{X}$ of ideals in R, there is an element $I \in \mathcal{X}$ which is not contained in any other element of $\mathcal{X}$.
- $2(a)$ For any $n \geq 0$ and any chain $M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots$ of submodules of R^n, we have $M_r = M_{r+1}$ for all sufficiently large r.
- $2(b)$ For any $n \geq 0$, every submodule M of R^n is finitely generated.
- $2(c)$ For any set $\mathcal{X}$ of submodules of R^n, there is an element $M \in \mathcal{X}$ which is not contained in any other element of $\mathcal{X}$.

  Let S be a finitely generated R-module:
- $3(a)$ For any chain $M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots$ of submodules of S, we have $M_r = M_{r+1}$ for all sufficiently large r.
- $3(b)$ Every submodule M of S is finitely generated.
- $3(c)$ For any set $\mathcal{X}$ of submodules of S, there is an element $M \in \mathcal{X}$ which is not contained in any other element of $\mathcal{X}$.

Problem 1 Complete the following exercises for your favorite choice of $*$ among $\{a, b, c\}$.

(a) Show that $3(*) \implies 2(*) \implies 1(*)$ for $* \in \{a, b, c\}$.

(b) Show that $2(*) \implies 3(*)$ for $* \in \{a, b, c\}$. . Hint: Take a surjection $\gamma : R^n \to M$, and take the preimages of the various objects under γ.

(c) Show that $1(*) \implies (2*)$. Hint: Use induction on n. For $n > 1$, look at the short exact sequence $0 \to R \xrightarrow{\iota} R^n \xrightarrow{\pi} R^{n-1} \to 0$. Given $M \subset R^n$, think about the modules $M \cap \iota(R)$ and $\pi(M)$. Remark: If R is a field, then $1(b)$ is obvious, but $2(b)$ is the first significant theorem in a linear algebra course. So you should expect to need to do some work here.

Problem 2 Complete the following exercises for your favorite choice of # among $\{1, 2, 3\}$.

(a) Show that $\#(b) \implies \#(a)$.

(b) Show that $\#(c) \implies \#(b)$.

(c) Show that $\#(a) \implies \#(c)$.

Remark: The implication $\#(a) \implies \#(c)$ requires the Axiom of Choice, although it does so in such a simple way that those of you not used to watching for such things may miss it. I believe (but am not certain), that without AC, the concepts (a), (b) and (c) are logically distinct. The other parts of this problem set do not require AC.

Problem 3 Show that a quotient ring of a noetherian ring is noetherian. Hint: This is easiest with the $3(*)$ properties.

Problem 4 We will now prove the **Hilbert basis theorem**: If A is noetherian, then $A[t]$ is noetherian. Hence, by induction on n, $k[t_1, t_2, \dots, t_n]$ is noetherian. Applying problem 3, this shows any finitely generated k-algebra is noetherian.

We will be proving $A[t]$ obeys $1(b)$. Let I be an ideal of $A[t]$. Define I_d to be the set of $g \in A$ such that there is an element of I of the form $gt^d + f_{d-1}t^{d-1} + \cdots f_1 t + f_0$.

(a) Show that I_d is an ideal of A.

(b) Show that $I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots$.

Using property $1(a)$ of A, there there is some ideal I_∞ of A so that $I_r = I_{r+1} = \cdots = I_\infty$. Using property $1(b)$ of A, take a finite list of generators g_1, g_2, …, g_k of I_∞. For each g_i, choose $f_i \in I$ of the form $g_i t^r +$ lower order terms.

(c) Show that $I \cap A \cdot \{1, t, t^2, \dots, t^{r-1}\}$ is finitely generated as an A-module. (Hint: Property $2(b)$ is your friend.)

Let h_1, h_2, …, h_ℓ be a list of generators for $I \cap A \cdot \{1, t, t^2, \dots, t^{r-1}\}$.

(d) Show that f_1, f_2, …, f_k, h_1, h_2, …, h_ℓ generate I as an $A[t]$ module.

Problem 5 This is the original purpose for which Hilbert proved his Basis Theorem. This is even more optional than the rest of the problem set, but it is really fun.

Let K be a compact group and let $\rho : K \to \mathrm{GL}_n(\mathbb{C})$ be a continuous representation. A polynomial f in $\mathbb{C}[x_1, \dots, x_n]$ is called ***invariant*** if $f(x) = f(g \cdot x)$ for all $g \in K$ and all $x \in \mathbb{C}^n$. Let S be the ring of invariant polynomials. We will show that S is finitely generated as a $\mathbb{C}$-algebra.

You will need to know that, in this context, it makes sense to integrate continuous functions over K. We write such an integral as $\int_{g \in K} f(g)$ and define $\mathrm{Vol}(K) = \int_{g \in K} 1$. You may use any true plausible property of such integrals.

Given $f \in \mathbb{C}[x_1, \dots, x_n]$, we write f as $f_0 + f_1 + \cdots + f_N$ where f_d is the sum of the degree d monomials in f.

(a) Show that f is in S if and only if each f_d is in S.

We'll write S_d for the polynomials in S which are homogenous of degree d.

(b) Let J be the ideal in $\mathbb{C}[x_1, \dots, x_n]$ which is generated by $\bigcup_{d \geq 1} S_d$. Show that there are finitely many invariants, s_1, s_2, …, s_r in $\bigcup_{d \geq 1} S_d$ which generate J as an $\mathbb{C}[x_1, \dots, x_n]$ ideal. (This is not just quoting the Hilbert Basis Theorem, since you have to prove that you can take the generators in $\bigcup_{d \geq 1} S_d$, but it isn't much harder.)

(c) Let $t \in S_d$ for some $d \geq 1$. Show that t is in the S-ideal generated by s_1, …, s_r. (Extremely explicit hint: By construction, we have $t = \sum s_i f_i$ for some $f_i \in \mathbb{C}[x_1, \dots, x_n]$. Show that $t = \frac{1}{\mathrm{Vol}(K)} \sum s_i \int_K f_i(g \cdot (x_1, \dots, x_n))$.)

(d) Let $t \in S_d$ for some $d \geq 1$. Show that t is in the $\mathbb{C}$-algebra generated by the s_i, …, s_r. (Hint: Induction on d.)

Before Hilbert's result (1890), there was an industry of finding explicit generators for S for various specific K's and specific representations of them. Paul Gordan painstakingly worked out the case $K = SU(2)$ for all representations of K. When he heard that Hilbert could always prove that the invariant ring was finitely generated, yet could not write down the generators in any example, he exclaimed "Das ist nicht Mathematik. Das ist Theologie." (In fairness to Gordan, one should note that he later became a major proponent and developer of Hilbert's ideas.)