Problem Set 1 – due September 10

See the course website for policy on collaboration.

This problem set contains a trick question!

Notation: The letter k denotes an algebraically closed field. For $X \subseteq k^n$, we write $I(X)$ for the set of polynomials in $k[x_1, \ldots, x_n]$ which vanish on all points of X. For $S \subseteq k[x_1, \ldots, x_n]$, we write $Z(S)$ for the set of points $(a_1, \ldots, a_n)$ on which all polynomials in S vanish. Given a finitely generated k-algebra A, we write $\mathrm{MaxSpec}(A)$ for the set $\mathrm{Hom}_{k-\mathrm{alg}}(A, k)$.

Problem 1 This problem checks some basic relations between the operations Z and I. Throughout this problem, X and Y denote an arbitrary subsets of k^n while S and T denote arbitrary subsets of $k[x_1, \ldots, x_n]$. These results come in pairs separated by a semicolon; **you need only prove one of each pair**.

(a) If $X \subseteq Y$, then $I(X) \supseteq I(Y)$; if $S \subseteq T$, then $Z(S) \supseteq Z(T)$.

(b) We have $Z(I(X)) \supseteq X$; we have $I(Z(S)) \supseteq S$.

(c) We have $Z(I(Z(X))) = Z(X)$; we have $I(Z(I(S))) = I(S)$.

Problem 2 We defined a subset X of k^n to be ***Zariski closed*** if X is of the form $Z(S)$ for some $S \subseteq k[x_1, \ldots, x_n]$ or, equivalently, if $X = Z(I(X))$. In this problem, we justify this terminology by verifying that the Zariski closed sets obey the axioms of a topology. Show that

(a) Any intersection of Zariski closed sets is Zariski closed.

(b) Any finite union of Zariski closed sets is Zariski closed.

(c) The sets $\emptyset$ and k^n are Zariski closed.

Problem 3 Let X be the curve $y^2 = x^3 + x^2$ in $\mathbb{A}^2$. We can parametrize X as $t \mapsto (t^2 - 1, t(t^2 - 1))$.

(a) What is the corresponding map $k[x, y]/\langle y^2 - x^3 - x^2 \rangle \to k[t]$?

(b) Assume k does not have characteristic 2. Show that the image of this map is $\{f \in k[t] : f(1) = f(-1)\}$.

Problem 4 Let $R = k[x, y]/(y^2 - x^3)$. Map $R \to k[t]$ by $x \mapsto t^2$ and $y \mapsto t^3$.

(a) Show that $\mathrm{MaxSpec}\, k[t] \to \mathrm{MaxSpec}\, R$ is bijective.

(b) Show that the inverse map is not regular.

Problem 5 Let $k[t] \times k[t]$ be the ring of ordered pairs $(f(t), g(t))$ of polynomials, with ring operations given by $(f_1, g_1) + (f_2, g_2) = (f_1 + f_2, g_1 + g_2)$ and $(f_1, g_1)(f_2, g_2) = (f_1 f_2, g_1 g_2)$.

(a) Write $k[t] \times k[t]$ as $k[x, y]/I$ for some ideal I. (Hint: I suggest making x correspond to (t, t) and y correspond to $(1, 0)$.)

(b) Give a bijection between $\mathrm{MaxSpec}\, k[t] \times k[t]$ and $k \sqcup k$.

(c) Map $k[t] \times k[t]$ to $k[t]$ by $(f, g) \mapsto f$. What is the corresponding map of MaxSpec's from k to $k \sqcup k$?

(d) Map $k[t]$ to $k[t] \times k[t]$ by $f \mapsto (f, 0)$. What is the corresponding map of MaxSpec's from $k \sqcup k$ to k?

Problem 6 Let R be a commutative ring and r an element of R. Show that $1 - rt$ is a unit of $R[t]$ if and only if r is nilpotent in R. (***Nilpotent*** means that $r^N = 0$ for some positive integer N.)

Problem 7 Let R be a commutative ring and let I be an ideal of R. Define $\sqrt{I}$ (called the ***radical*** of I) to be $\{f \in R :\ \exists_{N \in \mathbb{Z}_{\geq 0}}\ f^N \in I\}$.

(a) Show that $\sqrt{I}$ is an ideal.

(b) Show that $\sqrt{\sqrt{I}} = \sqrt{I}$.