Problem Set 10 – due November 19

See the course website for policy on collaboration.

Problem 1 Let $X = \operatorname{MaxSpec} A$ be an affine algebraic variety and let v be a vector field, meaning a regular map $v : X \to TX$ providing a section of the projection $TX \to X$. For f a regular function on A, we define $v(f)$ as follows: At each point $x_0 \in X$, take the class of $f - f(x_0)$ in $\mathfrak{m}_{x_0}$ and pair it with $v(x_0)$.

(a) Show that $v(f)$ is a regular function on X.

(b) Show that the map $v : A \to A$ obeys

$$v(f+g) = v(f) + v(g) \quad v(fg) = fv(g) + gv(f) \quad v(z) = 0 \text{ for } z \in k.$$

Such a map is called a k***-linear derivation***.

(c) Let $D : A \to A$ be a k-linear derivation. Show that there is a vector field $v : X \to TX$ giving rise to D.

Problem 2 Let X and Y be irreducible varieties of the same dimension, and let $\pi : Y \to X$ be a finite map. So $\operatorname{Frac}(Y)/\operatorname{Frac}(X)$ is an extension of fields. In this problem, we will explore what norm and trace mean geometrically.

Let x be a smooth point of X such that, for all $y \in \pi^{-1}(x)$, the map $\pi_* : T_yY \to T_xX$ is an isomorphism. We write $\pi^{-1}(x) = \{y_1, y_2, \ldots, y_n\}$. So n is both the scheme-theoretic and the naive length of $\pi^{-1}(x)$, and n must be the degree of π. In other words, $n = [\operatorname{Frac}(Y) : \operatorname{Frac}(X)]$.

(a) Show that we can find e_1, e_2, $\ldots$, e_n in $\operatorname{Frac}(Y)$ such that e_i is regular at all the points of $\pi^{-1}(x)$, with

$$e_i(y_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

(b) Show that the e_i generate $\bigoplus_i \mathcal{O}_{Y,y_i}$ as an $\mathcal{O}_{X,x}$ module. Conclude that the e_i are a basis for $\operatorname{Frac}(Y)$ over $\operatorname{Frac}(X)$.

(c) Let $f \in \operatorname{Frac}(Y)$ be regular at all the points of $\pi^{-1}(x)$. Show that $\operatorname{Tr}_{\operatorname{Frac}(Y)/\operatorname{Frac}(X)}(f)$ and $N_{\operatorname{Frac}(Y)/\operatorname{Frac}(X)}(f)$ are regular at x and

$$\left(\operatorname{Tr}_{\operatorname{Frac}(Y)/\operatorname{Frac}(X)}(f)\right)(x) = \sum_i f(y_i) \text{ and } \left(N_{\operatorname{Frac}(Y)/\operatorname{Frac}(X)}(f)\right)(x) = \prod_i f(y_i).$$

Problem 3 Let X be a smooth 1-dimensional variety, let x_0 be a point of X and let t be a regular function on x with $t \in \mathfrak{m}_x$ and dt a generator of $T^*_x X$.

(a) Let ω be a 1-form on $X \setminus \{x_0\}$. Show that ω can be uniquely written in the form

$$\omega = a_{-N}\frac{dt}{t^N} + a_{-N+1}\frac{dt}{t^{N-1}} + \cdots + a_{-1}\frac{dt}{t} + \eta \quad (*)$$

where η is a 1-form on X.

Define a_{-1} to be $\operatorname{res}_t \omega$. In this problem we will show that, if k has characteristic zero, then $\operatorname{res}\omega$ is well defined, independent of the choice of t. This is also true in characteristic p, but requires a more difficult proof. Let u be another element of $\mathfrak{m}_t$, which also generates $T^*_x X$.

(b) Let g be a regular function on $X \setminus \{x_0\}$. Show that $\operatorname{res}_u dg = 0$.

(c) In the notation of $(*)$, show that $\operatorname{res}_u a_{-i}\frac{dt}{t^i} = 0$ for $i \geq 2$. Show that $\operatorname{res}_u \eta = 0$. Show that $\operatorname{res}_u dt/t = 1$. Conclude that $\operatorname{res}_u \omega = \operatorname{res}_t \omega$.

Please turn over for one more problem.

Problem 4 In this problem, assume that k does not have characteristic 2. This problem explores an important standard example, the ***hyperelliptic curve***.

Let $a_{2h}x^{2h} + a_{2h-1}x^{2h-1} + \cdots + a_0$ be a polynomial without repeated roots and assume that a_{2h} and a_{2h-1} are not both 0. Let H_0 be the curve $y^2 = a_{2h}x^{2h} + \cdots + a_1 h + a_0$ in $\mathbb{A}^2$.

(a) Verify that H_0 is smooth.

(b) Let $\bar{H}_0$ be the closure of H_0 in $\mathbb{P}^2$. Show that $\bar{H}_0$ is *not* smooth, except when $h = 1$ or when $h = 2$ and $a_{2h} = 0$.

We will now construct a better compactification of H_0, which is smooth. Let H_∞ be the curve $v^2 = a_0 u^{2h} + a_1 u^{2h-1} + \cdots + a_{2h}$.

(c) Give an explicit isomorphism between $H_0 \setminus \{x = 0\}$ and $H_\infty \setminus \{u = 0\}$.

Map H_0 into $\mathbb{P}^{h+1}$ by $\phi(x, y) = (1 : x : x^2 : \cdots : x^h : y)$. Label the coordinates on $\mathbb{P}^{h+1}$ as $(z_0 : z_1 : z_2 : \cdots : z_h : z_{h+1})$. Let H be the closure of $\phi(H_0)$ in $\mathbb{P}^{h+1}$.

(d) Show that $H \cap \{z_0 \neq 0\} \cong H_0$ and $H \cap \{z_h \neq 0\} \cong H_\infty$.

(e) Let $\bigoplus S_d$ be the homogenous coordinate ring of H in $\mathbb{P}^{h+1}$. Show that S_d is isomorphic to the subspace of $\mathcal{O}_{H_0}$ spanned by $x^i y^j$ for $i/h + j \leq d$. Compute the dimension of S_d.

Let ω be the 1-form $\frac{dx}{2y}$ on $H_0 \setminus \{y = 0\}$.

(f) Show that ω extends to a regular 1-form on H_0. Show that every differential form on H_0 can be written uniquely in the form $(f(x) + g(x)y)\omega$ for some polynomials f and g.

(g) Show that the vector space of 1-forms on H_0 that extend to H has dimension $h - 1$.