Problem Set 11 – due November 26

See the course website for policy on collaboration.

This is the final problem set! I want you to have time after break to work on your papers and assimilate all that you have done.

Definitions/Notation A ***curve*** is a one dimensional variety. If X is a smooth curve and x is a point of X, then $\mathcal{O}_{X,x}$ is a regular local ring, and hence a dvr. We write $\mathfrak{m}_x$ for the maximal ideal of $\mathcal{O}_{X,x}$. An element $u_x \in \mathcal{O}_{X,x}$ is called a ***uniformizer at*** x if u_x generates $\mathfrak{m}_x$. We define the ***valuation*** v_x to be the map $\mathrm{Frac}(X) \setminus \{0\} \to \mathbb{Z}$ so that $f u_x^{-v_x(f)}$ is a unit of $\mathcal{O}_{X,x}$. We formally define $v_x(0) = \infty$. For X a smooth curve, we write $\mathrm{Frac}\,\Omega(X)$ for the space of rational 1-forms on X. If $x \in X$ and u_x is a uniformizer at X, and $\omega \in \mathrm{Frac}\,\Omega(X)$, then $\omega = f du$ for some $f \in \mathrm{Frac}(X)$, and we define $v_x(\omega)$ to be $v_x(f)$; this definition does not depend on the choice of u_x. Note that a function f (respectively, a 1-form ω) is regular at x if and only if $v_x(f)$ (respectively $v_x(\omega)$) is ≥ 0.

If X and Y are smooth curves, $\pi : Y \to X$ is a finite map, and $\pi(y) = x$, then the ***ramification of*** π ***at*** y is the positive integer e_y defined by either of the following conditions: (1) $v_y(\pi^*(u_x)) = e_y$, where u_x is a uniformizer at x or (2) $e_y = \dim \mathcal{O}_{Y,y}/\mathfrak{m}_x \mathcal{O}_{Y,y}$.

If X is a smooth projective curve, the ***genus*** of X, written g_X, is defined by $2g_X - 2 = \sum_{x \in X} v_x(\omega)$ for any $\omega \in \mathrm{Frac}\,\Omega(X) \setminus \{0\}$. The genus is always a nonnegative integer, but we don't have the tools to prove that yet. The ***Riemann-Hurwitz formula*** states that, in characteristic zero, we have

$$2g_Y - 2 = (\deg \pi)(2g_X - 2) + \sum_{y \in Y} (e_y - 1).$$

We recall the definition of the ***residue*** of a 1-form from the previous problem set: If X is a smooth curve, x is a point of X, the function u_x is a uniformizer at x and ω is a rational 1-form on X, then $\mathrm{res}_x \omega$ is the unique constant a_{-1} so that ω is of the form

$$\omega = \frac{a_{-N} du_x}{u_x^N} + \cdots + \frac{a_{-2} du_x}{u_x^2} + \frac{a_{-1} du_x}{u_x} + \eta$$

for η regular at x.

Problem 1 In the previous problem set, we constructed the ***hyperelliptic curve***: A smooth projective curve covered by the two affine curves

$$\begin{aligned} H_0 &= \{(x,y) : y^2 = a_{2h} x^{2h} + a_{2h-1} x^{2h-1} + \cdots + a_1 h + a_0\} \\ H_\infty &= \{(u,v) : v^2 = a_{2h} u^{2h} + a_{2h-1} u^{2h-1} + \cdots + a_1 u + a_0\} \end{aligned}$$

overlapping along $H_0 \setminus \{x = 0\} \cong H_\infty \setminus \{u = 0\}$. I remind you that the isomorphism between these charts is given by $u = x^{-1}$, $v = x^{-h} y$.

Compute the dimension (as a k vector space) of the cokernel of the map

$$\partial : \mathcal{O}_{H_0} \oplus \mathcal{O}_{H_\infty} \longrightarrow \mathcal{O}_{H_0 \cap H_\infty} \text{ where } \partial(f,g) = f|_{H_0 \cap H_\infty} - g|_{H_0 \cap H_\infty}.$$

Problem 2 Let $F(x,y,z)$ be a nonzero homogenous equation of degree d. Let $X \subset \mathbb{P}^2$ be the curve $F = 0$, and assume that X is smooth. Assume that $F = 0$ is transverse to $z = 0$ and $(\partial F/\partial x) = 0$, and that $\{F = \partial F/\partial x = z = 0\} \subset \mathbb{P}^2$ is empty.

(a) Let ω be the 1-form $d(x/z)$ on $\mathbb{P}^2$. Show that the restriction of ω to X has double poles (i.e. $v(\omega) = -1$) at the d points $X \cap \{z = 0\}$ and no other poles.

(b) Show that ω has simple zeros (i.e. $v(\omega) = 1$) at the $d(d-1)$ points $F = \partial F/\partial x = 0$, and no other zeroes.

(c) Compute the genus of X.

(d) Show that, for any degree $d-3$ polynomial $H(x,y,z)$, the 1-form $\omega \frac{z^2 H}{\partial F/\partial x}$ is regular on X.

Problem 3 Show that, if ω is a rational 1-form on $\mathbb{P}^1$, then

$$\sum_{x \in X} \mathrm{res}_x \omega = 0.$$

(Hint: partial fraction decomposition.)

Problem 4 In this problem, you may assume that the genus of any curve is a nonnegative integer.

(a) Let k have characteristic zero. Let X be a smooth projective curve and let $\pi : X \to \mathbb{P}^1$ be a map which is unramified over $\mathbb{P}^1 \setminus \{\infty\}$. Show that $g_X = 0$ and $\deg \pi = 1$.

(b) Show that problem (a) is false in characteristic p, by showing that the map $t \mapsto t^p - t$ from $\mathbb{A}^1 \to \mathbb{A}^1$ extends to a degree p map $\mathbb{P}^1 \to \mathbb{P}^1$ which is unramified above $\mathbb{P}^1 \setminus \{\infty\}$.

Problem 5 Let X and Y be smooth irreducible varieties of dimension d and let $\pi : Y \to X$ be a finite separable map of degree n. In this problem, we will define a ***trace map*** Tr from rational 1-forms on Y to rational 1-forms on X.

Let ω be a 1-form on Y, which we can also consider as a regular function on TY. Let $\mathrm{Tr}\, \omega$ be the rational function on TX defined by the trace map from $\mathrm{Frac}(TY)$ to $\mathrm{Frac}(TX)$.

We will say that π is ***unramified*** over a point $x \in X$ if, for all $y \in \pi^{-1}(x)$, the maps $\pi_* : T_y Y \to T_x X$ are isomorphisms. (When $\dim Y = \dim X = 1$, this is the same as all ramification degrees being 1.) We showed in class that there is a nonempty open U in X over which π is unramified.

(a) Show that TY and TX are smooth. Show that $\pi_* : TY \to TX$ is unramified over TU.

(b) Show that $\mathrm{Tr}\, \omega$ is regular on TU, and is a 1-form. (Hint: Problem 2, Problem Set 10.)

(c) Too many abstract definitions! Let k have characteristic $\neq 2$, let X and Y be $\mathbb{P}^1$ with coordinates u and v respectively, and let $\pi(v) = v^2$. Show that $\mathrm{Tr}\, dv/v = du/u$. (No, there should NOT be a coefficient of 2 or 1/2.)

(d) Let $f \in \mathrm{Frac}(Y)$ and let $\omega \in \mathrm{Frac}\, \Omega(X)$. Show that $\mathrm{Tr}(f \pi^* \omega) = \mathrm{Tr}(f) \omega$ where $\mathrm{Tr}(f)$ is the trace map $\mathrm{Frac}(Y) \to \mathrm{Frac}(X)$. Hint for parts (d) and (e): Since TX is irreducible, it is enough to verify this equality on TU.

(e) Let $g \in \mathrm{Frac}(Y)$. Show that $\mathrm{Tr}(dg) = d\, \mathrm{Tr}(g)$ where, on the right, Tr is the trace map $\mathrm{Frac}(Y) \to \mathrm{Frac}(X)$.

(f) Let k have characteristic zero and let $\dim X = \dim Y = 1$. Let $\eta \in \mathrm{Frac}\, \Omega Y$. Show that, for $x \in X$,

$$\mathrm{res}_x \eta = \sum_{y \in \pi^{-1}(x)} \mathrm{res}_x \omega.$$

Here the sum is **not** taken with multiplicity. Hint: choose a uniformizer t at x. Show that you can write ω in the form $f \pi^*(dt/t) + dg$ with f regular at z, and use parts (e) and (f).

Remark: Parts (b) and (f) of this problem are short of the full truth. In part (b), if ω is regular on Y then $\mathrm{Tr}\, \omega$ is regular on all of X, not just U. It is hard to find a reference for this; see the discussion at `http://mathoverflow.net/questions/167656`. In particular, Zannier *A note on traces of differential forms* (1999) claims to provide an elementary proof when X is normal; but the result is not true in that generality as de Jong and Starr *Erratum: Cubic fourfolds and spaces of rational curves* (2008) point out. I think that Zannier's proof is correct if we assume that X is smooth. Part (f) holds in any characteristic, not just 0. The standard reference is Serre, *Algebraic Groups and Class Fields*, Chapter II.13.